Financial Models and Portfolio Optimizations in the U.S. Market

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Abstract. Portfolio optimization is an essential task for investors to maximize the profit and minimize the risk in trading, which can be reflected in return and standard deviation respectively. This article selects ten well-known firms and S&P 500 index as the portfolio and calculates their basic descriptive data of annualized return, annualized standard deviation, and correlations. Next, this article applies Markowitz model and Single-index model to the portfolio optimizations, as well as setting five constraints for comparison. After using the above methods, this paper gives the results in tabular and graphic ways. It finds that optimizing with no restriction can generate the biggest permissible regions of the minimum-variance frontier; investors only long stocks under constraint 4 can guarantee a finite risk and a finite positive return, which has a robust performance. Moreover, Markowitz model is flexible in covariances, but the number of estimates could complicate the calculating process; Index model is simple to calculate in linear regression form, but it's not highly sensitive to the covariance matrix. The purpose of this article is to prove the feasibility of optimization in future portfolio management and helps investors to construct the portfolios with the fundamental theory of financial models.

Keywords: Portfolio optimization; Markowitz Model; Single-index model; Sharpe ratio.

1. Introduction

The financial market is complex, changeable, and uncertain. Over the years, investors have been seeking for the purpose to maximize returns and minimize risks. The best way to spread risks is to buy a basket of stocks instead of just one stock, which is called a "portfolio". After choosing the stocks, it is necessary to optimize the asset portfolio. The optimization method is usually to change the weight of each stock regularly through indicators such as the performance and yield of the stock. Thus, this paper will study two seminal models in the financial field.

Harry Markowitz formed the Markowitz Model firstly in 1952, and the assumptions of his model are all investors choose a variety of assets instead of just one asset, which is regarded as a classical technique. After the first introduction, he constantly updated his strategy over a few years. In 1959, Markowitz argued that under some conditions, choosing a portfolio from the mean-variance efficient frontier can maximize the investor's expected utility [1-2]. In this method, there should be some decision rules, and then attempt to get as close to the optimal weighting as possible, subject to the constraints. As this paper will explain later, Markowitz model uses covariance matrix in calculation to construct the portfolio, in this case, it needs to calculate the correlation coefficients between each stock in the portfolio.

William Sharpe developed the Single-Index Model in 1963 which contributed to the financial area until today. Sharpe got a hint from Markowitz and then formed a simple one with beta and alpha instead of covariances in Markowitz Model. Then, Sharpe earned the Nobel Prize in 1990 and contributed an essential theory in the investment field [3-4]. And this article sets 5 different constraints for portfolio optimization. Under these constraints, comparing the differences between Markowitz Model and Index Model, as well as the differences among each constraint, this paper will illustrate the advantages and drawbacks of each model, which can have a practical application in the finance area.

In this article, it selects 10 representative stocks in the U.S. market, and the purpose of this portfolio is to test two models in a best-fitting way. Besides these two models, this paper will focus on Minimal Variance and Sharpe Ratio in evaluating the performance of the portfolio. Investors prefer the minimal variance and the maximal Sharpe ratio when comparing a trade. These two models will also plot the minimum-variance frontier and find the permissible regions.

This paper is constructed as follows. Section 2 includes the data this article uses. Section 3 illustrates the methods for comparing the portfolio. Section 4 explains the results of the methods. Section 5 summarizes the whole paper and gives conclusions.

2. Data

In this article, the stocks' historical data is derived from Yahoo Finance (https://finance.yahoo.com). This website offers stock data, original content, and released news for investors, which allows us to analyze the financial market accurately and objectively. This paper selects ten well-known corporations in various sectors, such as consumer, technology, energy, and financial services. The purpose of this setting is to expand the research scope in various financial fields and reduce the correlations between each two of them. In order to calculate the descriptive data and compare these 10 stocks in a certain period, this paper sets the time ranging from 05/11/2001 to 05/12/2021, and then transfer the data from daily closing prices to monthly closing prices.

Ten stocks are PepsiCo, Inc. (PEP), Oracle Corporation (ORCL), United Parcel Service, Inc. (UPS), Adobe Inc. (ADBE), NVIDIA Corporation (NVDA), Microsoft Corporation (MSFT), Cisco Systems, Inc. (CSCO), Amazon.com, Inc. (AMZN), Wells Fargo & Company, and Chevron Corporation (CVX). In the analysis, this article also includes the S&P 500 index (SPX) in the portfolio and uses FEDL01 as the risk-free rate.

To fully apply the Markowitz Model and Single-Index Model to the portfolio, it needs to calculate several indicators. Annualized average return, annualized standard deviation, beta coefficient, annualized alpha, and residual standard deviation are shown in Table 1. Here, beta is the coefficient of the regression equation of the index model, and alpha is the intercept of the regression equation of the index model. Correlation Coefficients are shown in Table 2, which indicate associations between different variables. If the correlations approach to 1 or -1, it indicates a strong correlation, and the correlations approach to 0 indicates a weak correlation.

Table 1. Statistic Data Description

	SPX	PEP	ORCL	UPS	ADBE	NVDA	MSFT	CSCO	AMZN	WFC	CVX
Annualized Average Return	7.542%	7.888%	11.114%	9.850%	19.578%	32.802%	13.147%	9.714%	33.796%	8.888%	8.787%
Annualized Standard Deviation	14.850%	15.074%	27.817%	21.437%	31.790%	55.774%	23.299%	30.809%	41.410%	28.133%	22.307%
Beta	1	0.530	1.022	0.830	1.423	1.979	1.002	1.321	1.351	1.052	0.921
Annualized Alpha	0.000%	3.890%	3.406%	3.592%	8.847%	17.877%	5.588%	-0.246%	23.604%	0.954%	1.843%
Residual Standard Deviation	0.000%	12.856%	23.312%	17.543%	23.753%	47.405%	17.927%	23.762%	36.223%	23.397%	17.626%

Table 2. Correlation Coefficients of the Portfolio

	SPX	PEP	ORCL	UPS	ADBE	NVDA	MSFT	CSCO	AMZN	WFC	CVX
SPX	1	0.522	0.546	0.575	0.665	0.527	0.639	0.637	0.485	0.555	0.613
PEP	0.522	1	0.205	0.307	0.330	0.157	0.334	0.246	0.222	0.281	0.272
ORCL	0.546	0.205	1	0.320	0.445	0.383	0.475	0.448	0.336	0.249	0.264
UPS	0.575	0.307	0.320	1	0.396	0.250	0.341	0.353	0.296	0.415	0.307
ADBE	0.665	0.330	0.445	0.396	1	0.499	0.508	0.493	0.459	0.298	0.262
NVDA	0.527	0.157	0.383	0.250	0.499	1	0.394	0.487	0.377	0.073	0.294
MSFT	0.639	0.334	0.475	0.341	0.508	0.394	1	0.480	0.401	0.287	0.339
CSCO	0.637	0.246	0.448	0.353	0.493	0.487	0.480	1	0.437	0.254	0.332
AMZN	0.485	0.222	0.336	0.296	0.459	0.377	0.401	0.437	1	0.105	0.155
WFC	0.555	0.281	0.249	0.415	0.298	0.073	0.287	0.254	0.105	1	0.335
CVX	0.613	0.272	0.264	0.307	0.262	0.294	0.339	0.332	0.155	0.335	1

After comparing the indicators in Table 1, this paper observes that the maximum and minimum annualized average returns are AMZN (33.796%) and SPX (7.542%). The maximum and minimum annualized standard deviations are NVDA (55.774%) and SPX (12.850%). Also, only CSCO has a negative annualized alpha. In Table 2, ADBE and SPX have the highest correlation (0.665), and WFC and NVDA have the lowest correlation (0.073).

3. Methods

3.1 Markowitz Mean-Variance Portfolio Model

Harry Markowitz formed the Mean-Variance Portfolio Theory, also called the Modern Portfolio Theory. The assumptions of this model are all investors are risk-averse, which means that they are willing to avoid all unnecessary risks [5].

All risk-return opportunities could be reflected on the minimum-variance frontier. The upward of this frontier is called the efficient frontier of risky assets, and the bottom section of this frontier is called the inefficient frontier. Where the midpoint of these two parts lies on the horizontal line called the global minimum-variance portfolio, which stands for the minimum variance of all possible portfolios.

The sum of each weight is always equal to 1 since the model subjects to change different weights to optimize the portfolio as:

$$\sum_i w_i = 1 \tag{1}$$

Where w_i is each weight of the stock i in the portfolio.

The return of the portfolio is the sum product of each stock with its weight as:

$$R(P) = E(\sum_{i=1}^{n} w_i * r_i) \tag{2}$$

Where r_i is each return of the stock i.

The variance of the portfolio is as:

$$var = \sum_{i=1}^{m} \sum_{j=1}^{n} \left(w_i * \sigma_i * cor_{ij} * w_j * \sigma_j\right) \tag{3}$$

Where w_i and w_j are weights of two different stocks, σ_i and σ_j are their standard deviation respectively, cor_{ij} is the correlation between them. Also, if two portfolios have the same standard deviation, investors prefer a higher return.

Mean-Variance Model clarifies the risk-return assets, then identifies the optimal result of the portfolio's weights, and draws the steepest CAL, here, the slope of the CAL [6] is the Sharpe ratio:

$$S_p = \frac{E(r_p) - r_f}{\sigma_P} \tag{4}$$

Where $E(r_p)$ represents the return of the portfolio; r_f is the risk-free rate; σ_P is the standard deviation of the portfolio. In the same portfolio, investors prefer a higher Sharpe ratio since it can provide a higher return in a certain increasing risk [7].

3.2 Single-Index Model

William Sharpe simplifies the analysis by assuming there is only one systematic factor that affects stock return, and this factor could be plausibly represented by the rate of return of a market index, such as the S&P 500 index (SPX) [8]. This method leads to a new model which is called the Single-index model. Generally, this index model can be expressed as the following regression equation [9]:

$$R_i = \alpha_i + \beta_i * R_M + e_i \tag{5}$$

Where M is the market index, $R_M = r_M - r_f$ stands for the excess return of the market index, $R_i = r_i - r_f$ stands for the excess return of each stock i; α_i is the intercept of this equation; β_i is the slope of this regression equation which measures the sensitivity of this security; e_i specifies the residual.

The variance of the portfolio in the index model is as following:

$$Var = \sum_{i=1}^{n}[(w_i * \beta_i) * \sigma_M]^2 + (w_i * Residue_i)^2 \tag{6}$$

Where σ_M is the standard deviation of the market index.

The return of the portfolio in the index model is as following:

$$R(P) = \sum_{i=1}^{n}[\,(w_i * \beta_i) * E(r_M) + \sum_{i=1}^{n} w_i * E(\alpha_i)\,] \tag{7}$$

3.3 Five Constraints

This paper sets five different constraints to test the optimal weights under two models. Each constraint represents a different trading strategy and leads to various ranges of rate of return and standard deviation. These constraints allow us to optimize the portfolio.

1) No constraint.

There is no constraint, which means that the investors can long or short in any combination.

2) $|w_i| \leq 1\ \forall i$;

Each weight of the stock i is bounded between -1 to 1, including.

3) $\sum_{i=1}^{11} |w_i| \leq 2\ \forall i$;

This constraint means that the sum of the absolute value of each weight is less than 2.

4) $w_i \geq 0\ \forall i$;

This constraint forces the investors can only long stocks and cannot short stocks.

5) $w_1 = 0$;

This constraint does not include the SPX index in the portfolio.

3.4 Solver Table

After setting 5 different constraints, it needs the solver table in Excel to calculate the points on the minimum-variance frontier by Markowitz Model. There are two special points for observing, which are the minimum risk portfolio and maximum Sharpe portfolio:

Minimal Risk:

$$\sigma(p) \to min \tag{8}$$

Max Sharpe:

$$\frac{r(p)}{\sigma(p)} \to max \tag{9}$$

For the minimum-variance frontier, efficient frontier, and inefficient frontier, it also needs to use the solver table to find the weights. After calculating all the possible points on the frontier, it can plot the permissible regions for opportunities. In this process, it sets a variable called “dummy variable”, which equates all constant terms with dummy variables [10]. Three parts of the frontier are as following:

Minimum-Variance Frontier:

$$\begin{cases} \sigma(p) \rightarrow min \\ subject\ to: \quad r(p) = \ dummy\ variable \end{cases} \tag{10}$$

Efficient Frontier:

$$\begin{cases} r(p) \rightarrow max \\ subject\ to: \quad \sigma(p) = \ dummy\ variable \end{cases} \tag{11}$$

Inefficient Frontier:

$$\begin{cases} r(p) \rightarrow min \\ subject\ to: \quad \sigma(p) = \ dummy\ variable \end{cases} \tag{12}$$

4. Results

Combining the above methods, this paper evaluates the performance of the portfolio under five different constraints by both Markowitz model and Index model. In order to compare two models and different constraints, this paper displays the result in both tabular and graphic ways for observation.

Under five constraints, the statistical data for evaluating the performance are rate of return, standard deviation, and Sharpe ratio. This paper will analyze two unique points on the efficient frontier, the solver table sets for minimum variance and maximum Sharpe ratio.

Constraint 1 is shown in Table 3, which has no restriction. Index model has a smaller standard deviation than Markowitz model, which is respectively 11.943% and 12.359%. Markowitz model has a higher Sharpe ratio than Index model, which is respective 1.124 and 1.108.

Table 3. Portfolio Performance under Constraint 1

	Stats / Min Variance	Return	StDev	Sharpe	Stats / Max Sharpe	Return	StDev	Sharpe
Constraint 1	MM	5.330%	12.359%	0.431	MM	36.220%	32.215%	1.124
	IM	3.832%	11.943%	0.321	IM	45.725%	41.258%	1.108

Constraint 2 is shown in Table 4, which means the portfolio does not include S&P 500 index. Index model has a smaller standard deviation than Markowitz model, which is respectively 11.943% and 12.359%. Markowitz model has a higher Sharpe ratio than Index model, which is respectively 1.092 and 1.079.

Table 4. Portfolio Performance under Constraint 2

	Stats / Min Variance	Return	StDev	Sharpe	Stats / Max Sharpe	Return	StDev	Sharpe
Constraint 2	MM	5.330%	12.359%	0.431	MM	24.699%	22.615%	1.092
	IM	3.832%	11.943%	0.321	IM	28.051%	25.988%	1.079

Constraint 3 is shown in Table 5, which means that the sum of the absolute value of each weight is less than 2. Index model has a smaller standard deviation than Markowitz model, which is respectively 11.943% and 12.359%. Index model has a higher Sharpe ratio than Markowitz model, which is respectively 1.039 and 1.015.

Table 5. Portfolio Performance under Constraint 3

	Stats / Min Variance	Return	StDev	Sharpe	Stats / Max Sharpe	Return	StDev	Sharpe
Constraint 3	MM	5.330%	12.359%	0.431	MM	24.292%	23.944%	1.015
	IM	3.832%	11.943%	0.321	IM	23.663%	22.770%	1.039

Constraint 4 is shown in Table 6, which means that the investors can only long stocks and cannot short stocks. Markowitz model has a smaller standard deviation than Index model, which is respectively 12.872% and 12.926%. Index model has a higher Sharpe ratio than Markowitz model, which is respectively 0.971 and 0.948.

Table 6. Portfolio Performance under Constraint 4

	Stats / Min Variance	Return	StDev	Sharpe	Stats / Max Sharpe	Return	StDev	Sharpe
Constraint 4	MM	8.134%	12.872%	0.632	MM	18.468%	19.483%	0.948
	IM	7.973%	12.926%	0.617	IM	21.220%	21.844%	0.971

Constraint 5 is shown in Table 7, which means that the investors can only long stocks and cannot short stocks. Markowitz model has a smaller standard deviation than Index model, which is respectively 12.967% and 12.991%. Index model and Markowitz model have the same Sharpe ratio which is 0.998.

Table 7. Portfolio Performance under Constraint 5

	Stats / Min Variance	Return	StDev	Sharpe	Stats / Max Sharpe	Return	StDev	Sharpe
Constraint 5	MM	8.003%	12.967%	0.617	MM	20.910%	20.960%	0.998
	IM	7.217%	12.991%	0.556	IM	23.277%	23.331%	0.998

Moving the focus from differences between Markowitz model and Index model to five constraints. This paper finds that constraints 1, 2, and 3 have the same global minimum-variance portfolio in both Markowitz model and Index model, which are respectively (12.359%, 5.330%) and (11.943%, 3.832%), noticing that the x-coordinate represents the standard deviation, and the y-coordinate represents the rate of return.

By plotting the minimum-variance frontier for Markowitz model and Index model in Fig. 1 and Fig. 2, this paper observes that they have a similar shape since both models are based on the same portfolio. In both models, constraint 1 has the biggest permissible region since no constraint could gain more possibilities of weights and opportunities for profit. Also, constraint 1 has the greatest maximum Sharpe ratio, thus giving the steepest CAL. For constraints 2, 3, and 5, they all converge well and could approach the infinite return or the infinite loss.

Besides, this paper wants to emphasize constraint 4, which plots a closed region in both models. The max efficient frontier is roughly equal to the max inefficient frontier. This is because constraint 4 can't short stocks, which will lead to a finite return and standard deviation. Moreover, the max annualized standard deviation in our portfolio is NVDA (55.774%), which decides the position of the closed point in two models, which is (55.5%, 32.8%). Although only long stocks could promise the finite rate of return, it could also promise a finite risk and give a positive return. Thus, if an investor is risk averse, this investor could choose constraint 4 to minimize risks and gain a certain return.

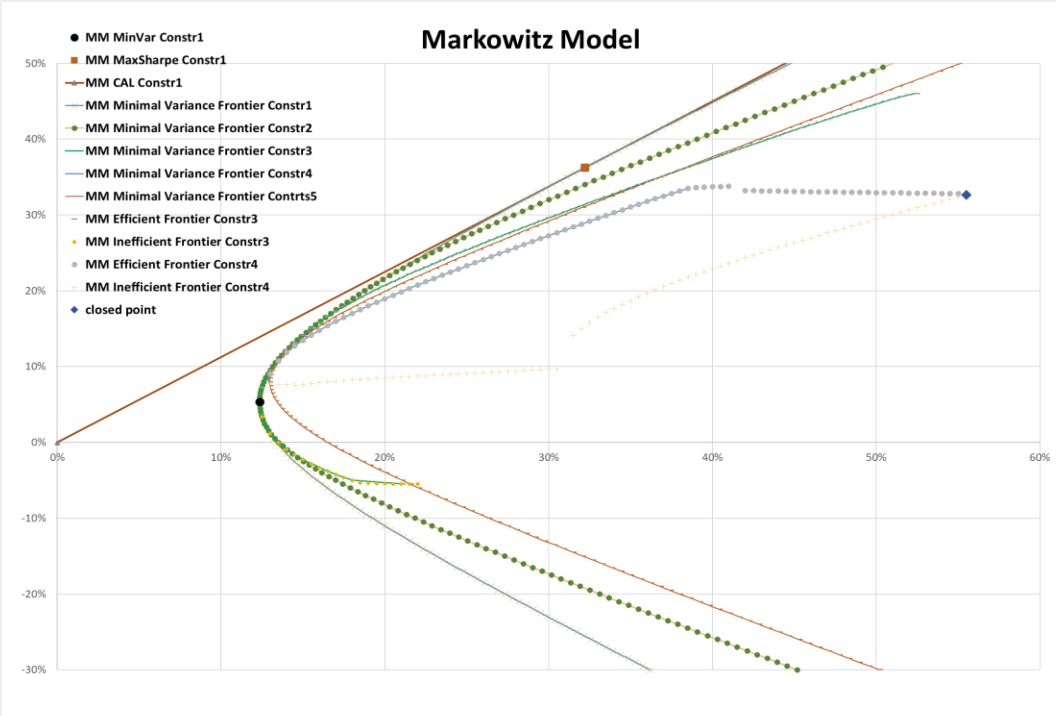

Fig. 1 Minimum-Variance Frontier for Markowitz Model

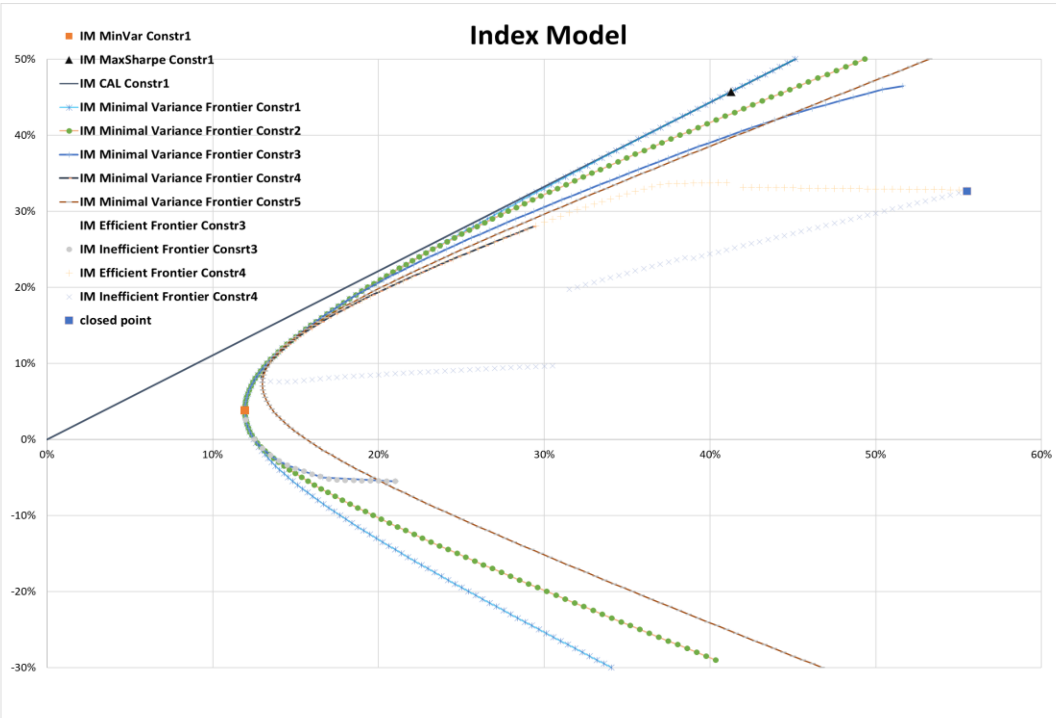

Fig. 2 Minimum-Variance Frontier for Index Model

5. Conclusion

This article chooses ten corporations and uses S&P 500 as the market index in the portfolio. Using stock data of twenty years from 2001 to 2021, which provides enough timeline for observing. Combining both Markowitz model and Single-index model, this paper uses the solver table in excel to generate the performance results of return, standard deviation, and Sharpe ratio. By comparing the tabular result, this paper finds that constraints 1, 2, and 3 have the same global minimum-variance portfolio point on the frontier, and Markowitz model and Index model can both give optimal results under different constraints. By comparing the graphic result, this paper finds that two models have roughly similar appearance and similar constraints results; constraint 1 which is a free constraint can attain the maximum Sharpe ratio among all constraints; constraint 4 means not short stocks can promise a positive return and a finite risk, which is the most robust performance. However, in order to calculate using Markowitz model, it needs to first calculate the covariances matrix for all asset which may need to deal with lots of estimates, and this process may be complicated and could cause some errors; notwithstanding, Markowitz model is more flexible for assets choosing since the correlation coefficient measures the strength of associations between stocks. As for Single-index model, it optimizes the process of factors, which is a linear regression with the systematic factor represented by the market index; this model only needs the slope and intercept of the linear equation which is simpler in the calculation, whereas the Index model is not highly responsive to the covariances.

Admittedly, the data in this paper includes a period of twenty years which may have an unpredictable “black swan” in the real financial market. For example, this period includes the financial crisis of 2007-2008, which may not have a more perfect result prediction. However, the financial model is ideal, and there will always be other factors in real transactions. Besides the optimization strategies analyzed in this paper, investors also need to assess the situation. The pandemic of Covid-19 in the recent three years has also led to economic stagnation to some extent, which may affect the smooth operation of the economic market. Therefore, this paper suggests that investors should combine the financial model with the actual financial market situation in constructing practical strategies.

To sum up, Markowitz model and Single-index model are still effective in today’s U.S. market, and their application could successfully help investors to increase the rate of return and decrease the risk. Two models and various constraints are indispensable for finding the optimal weights in the portfolio of the financial field. For future development, portfolio management and optimizing strategies still deserve attention and research.

References

[1] Markowitz H. Portfolio Theory: As I Still See It. Annual Review of Financial Economics, 2010, 2(1): 1–23.

[2] Zhang Y.P., Liu, Z.X., Yu X.Y. The diversification benefits of including carbon assets in financial portfolios. Sustainability, 2017.

[3] Mandal N. Sharpe’s single index model and its application to construct optimal portfolio: an empirical study. Great Lake Herald, 2013, 7(1): 1-19.

[4] Johnson B. M. The role of agribusiness assets in investment portfolios. Australasian Agribusiness Review, 2006, 14.

[5] Keizer J. Mean-Variance Portfolio Theory. CFA, FRM, and Actuarial Exams Study Notes, 2021.

[6] Bodie Z., Kane A., Marcus A. Investments - Standalone Book, McGraw Hill, 2017.

[7] Zakamouline, Koekebakker S. Portfolio performance evaluation with generalized Sharpe ratios: Beyond the mean and variance. Journal of Banking & Finance, 2009, 33(7): 1242–1254.

[8] Wikipedia. Available online at: https://en.wikipedia.org/wiki/Single-index_model (Accessed on 2022/11/15).

[9] Muslim. Return and Risk Comparative Analysis in the Formation of Optimal Share Portfolio with Random Model, Markowitz Model, and Single Index Model. Majalah Ilmiah Bijak, 2020, 17(2): 184–203.

[10] McFedries. Excel data analysis for dummies / by Paul McFedries. John Wiley & Sons, Inc, 2019.