Analyzing Investment Portfolios by Distinct Risk Preferences Using Single Index Model

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Abstract. At present, many investors perform irrationally in the stock market: they may invest more and short more shares when they lose money in the stock market and be less risky when gaining money. Different risk preference people also perform differently. This study would give investors who have distinct risk preferences investment restrictions so that investors' returns can be controlled if they invest under certain constraints. This study imports daily data of ten top stock and market index SPX500 and processes these data possible for the Single-Index Model to run. Based on this model, this study keeps original efficient and inefficient frontiers and adds four constraints to find the difference in curves and Sharpe ratio. As a result, this study compares data and curves in the above five situations and suggests risk-averse investors take "no short selling" constraints in their portfolio and risk-seeking investors take the no-constraint portfolio to optimize their profits in the ideal condition. Among four constraints, limiting the weight of each index to less than one is best for risk-seeking investors. Customizing investment portfolios under certain constraints by different risk preference levels using the index model is a good tool for investors to control their risk.

Keywords: Investment portfolios; Risk preference; Single index model; Sharpe ratio.

1. Introduction

1.1 Background

In portfolio allocation, the main objective is to combine empirical information on security prices with theoretical models to generate the optimal solution that best suits the investor's risk appetite.

Risk preference, which refers to people's attitudes towards risk, is a key factor in the study of investor decision-making behavior. The conventional financial theory makes the assumption that investors are logical, but their real-life decision-making behavior is less rational and is influenced by many factors such as behavioral science, psychology, and genetics. For example, people hate risk in times of gain and seek risk in times of loss. But in such cases, those who seek profit at a time of loss tend to lose more. For example, when people go to a casino to play Texas Hold'em, they tend to play bigger when they are losing money, they bet more chips on cards with smaller odds. In this case, investors need to be disciplined in their investment strategies when investing in the stock market, especially for irrational investors like those mentioned above. This way, they can know the expected and maximum, and minimum returns before they start investing and they can have a rational investment based on their risk preference. Without any restrictions on the investment strategy, even if an optimal solution is made by combining the investment components of each stock, the risk is high because the price of the stock will always change and the investor does not know what return they will get, which is likely to be a minimum return, i.e. a negative one. Restriction to short sell is a safe strategy for the risk-averse investor to use and reduce the risk because they can no longer borrow a security and all weights of stocks would be positive.

To tackle the problem, the single-index model (SIM) with constraints is applied in this study. It is a finical model developed by William Sharpe in 1963 [1]. The index model can measure the risk and the return of portfolios. The Single Index Model (SIM) identifies two sources of uncertainty in securities returns: systemic (macroeconomic) uncertainty (the assumption that is assumed that a single stock return index like SPX); microeconomic uncertainty (represented by security-specific stochastic factors). The return of any stock, according to this model, can be broken down into three categories: the return resulting from market-wide macroeconomic events, the unexpected microeconomic events that only affect the firm, and the expected excess return of the individual stock due to firm-specific factors, commonly indicated by its alpha coefficient. Specifically, the return of stock [2]:

$$ri = \alpha i + \beta irm + ei \tag{1}$$

The term $\boldsymbol{\beta_i r_m}$ refers to the stock's return because of market movement that has been modified ($\boldsymbol{\beta_i}$), which refers to the unsystematic risk associated with the security as a result of firm-specific traits. Based on this model, constraints like no-short can be applied to this model. The curve with max return concerning risk and the curve with min return concerning risk can be compared between constraints and free problem. According to those curves, investment portfolios for distinct risk preferences can be interrupted.

1.2 Related Research

Niranjan Mandalan attempts to gain knowledge of Sharpe's single index's underlying theory model and to experimentally build an ideal portfolio utilizing this methodology. The proposed method develops a distinct cut-off rate and chooses those securities to build an ideal portfolio whose excess return to beta ratio is higher than the cut-off rate. As a result, it is evident that using Sharpe's Single Index Model to formulate optimum portfolio investments is easier than using Markowitz Model [3]. J. Francis Mary and G. Rathika introduce how the index model is built, and each variable in this model such as beta, sharp ratio, and so on in this study. They use a constraint of non-short to compute an optimal portfolio and analyze it. According to the empirical investigation, only one business out of ten is chosen for investment purposes based on the Cut-off Point, which is -0.11182 [4]. The goal of Nalini R is to educate investors about the value of Sharpe's Single Index Model while constructing a portfolio. Since more firms are being traded on stock exchanges every year, Indian investors may also profit from Sharpe's Single Index Model (SIM). The fund managers of developing nations like India, where the markets are still growing, will benefit more from the findings of this study [5].

There are also other methods to formulate investment portfolios. In 2009, Thompson T H includes SAS software to compute the weights for at least a fifty-stock portfolio given a defined return or a series of returns. Only two or three stock portfolios are represented by the publically available SAS portfolios currently in existence. We also use risk aversion techniques to boost investor utility. Then he compared the result with Excel and conclude that the SAS program is faster and more accurate than the Excel spreadsheet [6]. In 2020, Thompson T H uses the SAS program to design an efficient investment portfolio with a constraint that no short sales. He displays the 50-stock portfolio's weights, with a weekly dividend of return by the SAS program and he adds a constraint that the utility is maximized. He proved the SAS program is accurate and computes the relationship between return and variance with the constraint of non-short sales [7]. This work uses the Particle Swarm Optimization (PSO) technology to propose a meta-heuristic solution to the portfolio optimization issue. The model is put to the test using a variety of limited and unconstrained hazardous investment portfolios, and a comparison study using genetic algorithms is carried out. In creating the best risky portfolios, the PSO model exhibits excellent computing efficiency. The technique is quite promising, according to preliminary data, and produces outcomes on par with or better than those of the most advanced solvers [8].

Risk preference and investment portfolios can be related to each other. Dietmar uses an Index tracking model which is concerned with forming a portfolio that mimics a benchmark index as closely as possible to analyze loss aversion when people are investing. And Dietmar uses a computational method to study e financial implications for the US stock. Additionally, they address various calibration problems for the optimization problem using differential Evolution [9]. Maller R A, Durand R B, and Jafarpour H discuss the role of the maximum Sharpe Ratio in determining the optimal portfolio choice. The first role is the best return-risk combination, and the second role is to adjust the returns based on the risk that has been taken after that. They import the distribution of Tn as an application in investment practice when calculating the maximum Sharpe Ratio [10]. Yu et al. use inverse optimization to measure risk preference by using two risk measurements —— The inverse of the Sharpe Ratio and Mutual Fund beta value. Then they use two cases study—– Robotic portfolio and Mutual Fund —– to create the relationship between risk tolerance and inverse sharp ratio. They successfully measure risk preference by those two risk measurements [11]. WEN et al. build a model by importing the top 10 stocks in Global Stock Exchange and dividing returns to gain and losses. In this model, they found for the same size of gains and losses, when losses happen, people tend to seek more risk when they have gained. In addition, when the magnitude of gains is increasing, investors' risk aversion improves [12].

1.3 Objective

This paper tends to use Single Index Model to help investors who have different risk preferences to customize their investment portfolios based on some constraints.

2. Method

2.1 Single Index Model (SIM)

To build this model, setting up, downloading, and organizing the data necessary for our computation and analysis are the initial steps in computing the SIM on Excel. This study imports the S&P 500 index ("SPX"), 10 stocks, and an instrument indicating a risk-free rate, a 30-day annual Fed Funds rate ("FEDL01"), which have all had daily total return data for 20 years. [13]. The S&P 500 index ("SPX") is the market index in this study, and the market capitalizations of the firms in the SPX are modified by the number of shares available for public trading because the index is float weighted. Ten stocks are Amazon.com, Inc. Apple Inc. Citrix Systems, Inc. JPMorgan Chase & Co. Berkshire Hathaway Inc. The Progressive Corporation United Parcel Service, Inc. FedEx Corporation, J.B. Hunt Transport Services, Inc., and Landstar System, Inc., belong to consumer cyclical, technology, financial services, and industrials. These ten notable stocks from four different industries and the S&P 500 index will be analyzed as investment allocations in this paper. The S&P 500 index ("SPX") will then be used as a free interest for each stock to calculate its access return, which will be covered in a later step.

The data needs to be processed after it is introduced before it can be applied to the model. First, the daily data are required to aggregate the monthly observations in order to lessen the non-Gaussian impacts. Then, by dividing the current month's price by the prior month's price and deducting one, get the return for each stock and SPX. The 1-month annual Fed Funds rate ("FEDL01") for the current month is then subtracted from the previous month's rate by one to calculate the free interest rate. FEDL01's return is a good representation of the free interest rate because it is stable without any risk. By subtraction, the excess return for each stock can be computed. After that, the average excess return, and the standard deviation for each can be computed as r_i and Std_i; these two indexes for SPX can be computed as r_M and Std_M. The significant measurement to return and risk, the Sharpe ratio is equal to excess return divided by standard deviation. The next step is to calculate the two most important coefficients in the index model, which are α_i and β_i. α_i is the intercept of the monthly returns of the SPX and the monthly returns of the stock and times 12 and β_i is the slope of the monthly returns of the SPX and the monthly returns of the stock. The last thing before entering the SIM is to calculate the residual excess and residual standard deviation. The residual return's formula is given here:

$$r_{residual} = r_i - r_M \times \beta - \alpha/12 \tag{2}$$

Where residual standard deviation, $residual_i$ can be calculated by residual returns.

So far, all the data needed for the index model has been processed, now let's calculate the parameters of the model. Before that, this model gives SPX and each stock the weight w_i and lets the sum of them equals 1. The return of this model:

$$R = E(sum(w_i \times \beta_i \times r_M + w_i \times \alpha_i + w_i \times residual_i)) = sum(w_i \times \beta_i) \times E(r_M) + sum(w_i \times E(\alpha_i) \quad (3)$$

and the variance of this model:

$$var(R) = var(sum(w_i \times \beta_i \times r_M + w_i \times \alpha_i + w_i \times residual_i))\text{=}(\, sum(w_i \times \beta_i) \times (r_M))^2 + (w_i \times residual_i)^2 \quad (4)$$

$Std(R) = \sqrt[2]{var(R)}$ and the Sharpe ratio of the index model is equal to $(R \div Std(R))$. Then each weight is randomly disrupted (but the sum is always equal to 1) to generate 1000 rows of data so that there are 1000 random investment portfolios and 1000 returns, standard deviations, and Sharpe ratios (as shown as portfolios in Figure 1). It uses Monte Carlo Simulation, a random number of generators method [13]. The next step is to go for two important points in this model. The first point is the point of minimum variance (as shown as MinVar in Figure 1), which is the point of minimum risk by continuously trying different weights on different stocks and SPX to find the point of minimum standard deviation. The second point is the point of maximum Sharpe ratio (as shown as MaxSharpe in Figure 1), also by continuously trying different weights on different stocks and SPX until the Sharpe ratio is maximum. This point means the best return-to-risk ratio and is the optimal point. Connecting the origin and this point and doubling it is the capital allocation line (as shown as CAL in Figure 1).

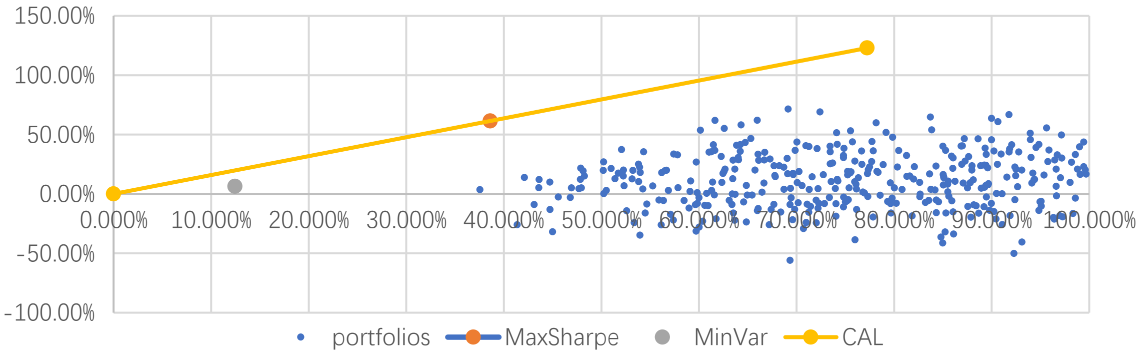

Fig. 1 Simulating Single Index Model

2.2 Add constraints

To help investors to build this investment plan, this paper provides a line of the highest returns as an efficient frontier and a line of the lowest as an inefficient frontier by drawing the standard deviation at 0.5 percent intervals and the corresponding max return/min return points and connecting them to a line. In this case, we can analyze the highest return, the lowest return, and their previous differences for a certain level of risk by looking at the curves on the image.

Before adding constraints to this model, a “free” problem, without any additional constraints need to be graphed, to show what the efficient frontier and the region of allowed portfolios would look like if there were no restrictions and to compare with the following figures with some constraints. The first constraint this paper applied to this model is no short so investors cannot obtain a security from their broker that they believe will decrease in value and sell it on the open market. In this case, each weight for each index must be larger or equal to zero when drawing the efficient and inefficient lines. The second constraint this paper applied in this model is designed to be an arbitrary “box” constraint on weights: $|w_i| \leq 1$. This is to allow investors to limit the size of their investments under the condition that shorting is allowed so that the absolute value of each stock does not exceed the total. In this case, each weight’s absolute value for each index must be less or equal to one when drawing the efficient and inefficient lines. The third constraint this paper applied in this model is designed to simulate Regulation T by FINRA. This enables broker-dealers to permit their clients to hold positions with at least 50% of their equity coming from their accounts: $\sum_{i=1}^{11}[w_i] \leq 2$. In this case, the sum of each weight's absolute value for each index must be less or equal to two when drawing the efficient and inefficient lines. The fourth constraint this paper applied in this model is the exclusion of the market index into our portfolio. In this case, the weight of the market index must be 0 when drawing the efficient and inefficient lines.

3. Result

3.1 Free Problem

As shown in Figure 2, the efficient frontier (Max return) starts from the min variance point and extends towards the capital allocation line, and slowly overlaps with it. An inefficient frontier (Min return) extends the line down at the same angle and symmetrically with the above line at the point of min variance.

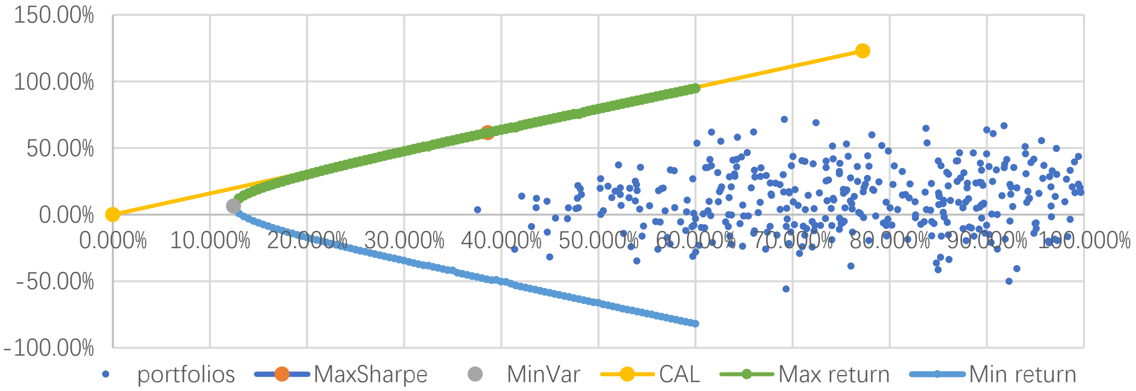

Fig. 2 Free problem

3.2 No short selling

As shown in Figure 3, the efficient line (Max return) does not start from the min variance point but starts from the point (13.500%, 13.500%). This line extends towards the capital allocation line, but it does not reach the capital allocation line. It finally stops at (32.35%, 27.00%). The inefficient line (Min return) also starts near the min variance point and moves slowly upward. Both two lines are short.

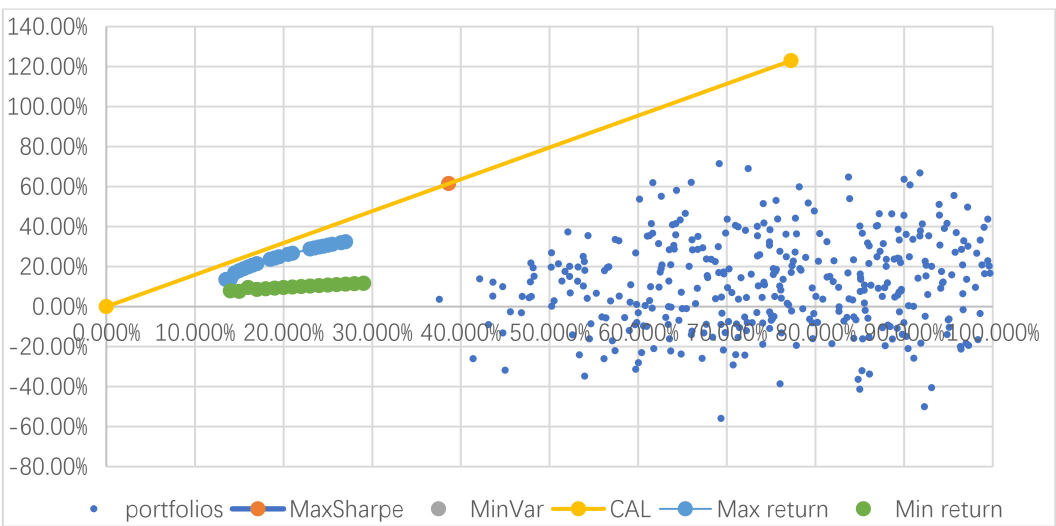

Fig. 3 No Short

3.3 Restrictions on individual stock shares

As shown in Figure 4, the efficient l frontier (Max return) starts from the min variance point and extends towards the capital allocation line, and is tangent with it. It does not pass the capital allocation line. The inefficient frontier (Min return) also starts from the min variance point, extending the line down at the same angle and symmetrically with the above line at the point of min variance. There are some outliers on these two curves.

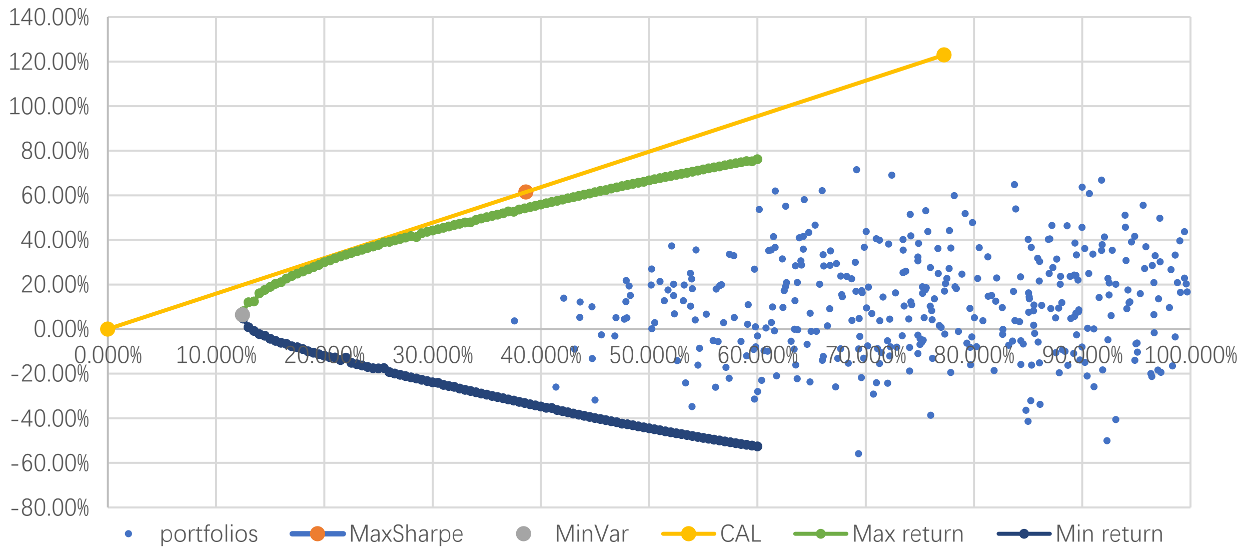

Fig. 4 Each weight <=1

3.4 Regulation T by FINRA

As shown in Figure 5, the size of the efficient frontier (Max return) is smaller and is tangent to the capital allocation line. The inefficient frontier (Min return) first goes down until the return is less than 0, then it is parallel to the x-axis. Then it suddenly goes up sharply and then up at a constant slope until it is almost level with the highest point of the efficient frontier. The gap between the two lines is gradually closing.

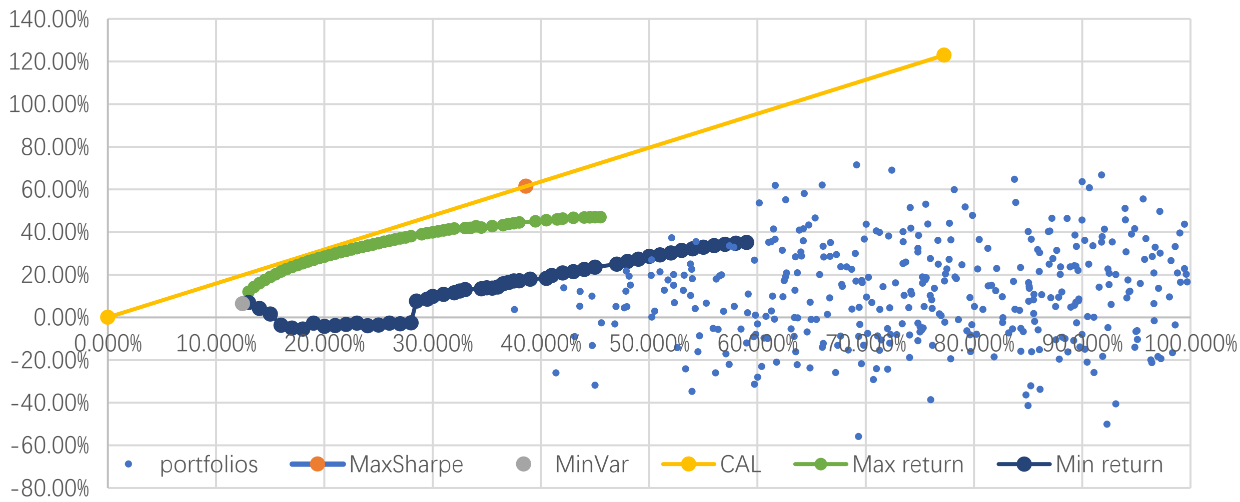

Fig. 5 Regulation T by FINRA

3.5 Exclusion of the market index

As shown in Figure 6, either of these frontiers passes through from the min variance point. the efficient line (Max return) is not tangent to the capital allocation line and basically goes all the way up with a fixed slope. The inefficient line (Min return) basically goes down with a fixed slope.

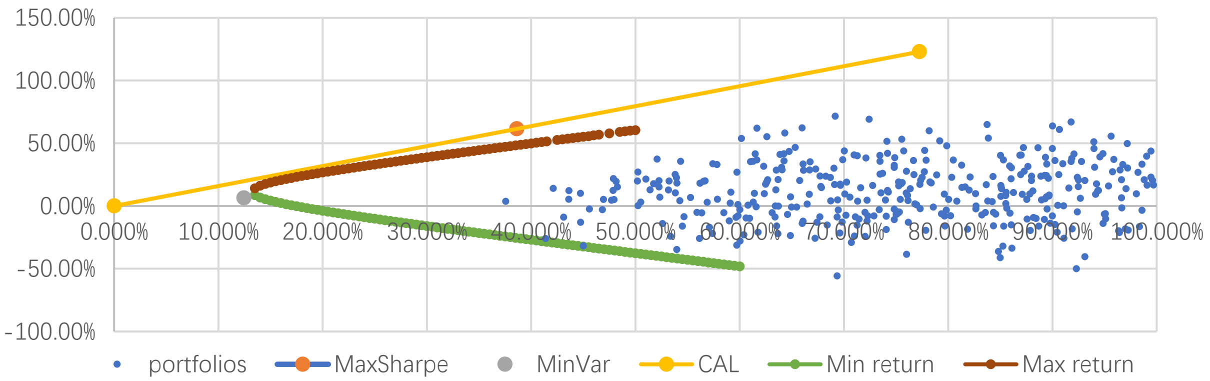

Fig. 6 Exclusion of Market Index

4. Discussion

4.1 Analysis of Combined Data

As shown in Table 1, obviously in "Free Problem", the max Sharpe ratio is the largest of the five because there is no limit to how large an investor can make the Sharpe Ratio. The Sharpe ratio is the return on investment with its risk [14]. In the free problem, the return is 58.09% in the case of risk is 36.50%. Although the Sharpe ratio and return are large, the risk is also not small, because they can invest as much as they want, as readers can see in the free problem SPX was shorted nearly three times the total investment ratio. However, in "No short selling", which is $|w_i| \geq 1$, the Sharpe ratio is smallest, because it cannot be shorted so each weight must be greater than 0 and add up to 1. This makes the risk and returns smaller because the size of the investment becomes smaller. This max Sharpe ratio is equal to 127.46% and the risk is equal to 19.50% when the return is equal to 24.85%. In this case, five of the ten stocks account for between thirteen and twenty-nine percent, one for three percent, and the other three and the SPX are all 0. This allocation reduces the risk at the same time as the Sharpe ratio is maximum. The max second constraint, $|w_i| \leq 1$, restricts the size of each index to less than 100% of the total. The Sharpe ratio is 151.76%, which is 7.39% smaller than in the free problem, but the return and risk are much smaller than the former. For the third constraint, the investor can borrow no more than 50% of the purchase price so that the size of short selling is restricted, resulting that the max Sharp ratio being 142.61%, which is 16.54% less than the max Sharpe ratio in the free problem. The last constraint is to exclude the market index, the Sharpe ratio is only 132.8% without any weights by market index.

Table 1. The weights comparison of different constraints

	Free Problem	No short selling	|w_i|≤1	Regulation T	SPX=0
SPX	-307.12%	0.00%	-100.00%	-47.75%	0.00%
AMZN	39.79%	15.30%	21.31%	17.09%	15.81%
AAPL	69.25%	29.66%	38.53%	31.78%	29.47%
CTXS	11.43%	0.00%	3.84%	0.74%	0.31%
JPM	6.40%	0.00%	-9.38%	-2.22%	-19.89%
BRK/A	57.82%	3.07%	34.17%	23.13%	16.91%
PGR	66.93%	19.54%	38.54%	30.14%	25.76%
UPS	26.38%	0.00%	9.51%	1.32%	-2.14%
FDX	22.74%	0.00%	5.38%	0.11%	-5.10%
JBHT	46.38%	13.35%	24.17%	18.66%	15.71%
LSTR	60.00%	19.06%	33.93%	27.01%	23.17%
Return	58.09%	24.85%	34.15%	28.52%	25.23%
StDev	36.50%	19.50%	22.50%	20.00%	19.00%
Sharpe	159.15%	127.46%	151.76%	142.61%	132.80%

4.2 For Risk-averse Investor

Risk-averse investors always tend to avoid risk and diversify their portfolios [15]. Thus, no short selling, Regulation T, and exclusion of Market Index fit with this standard because they have smaller standard deviation (risk), and a diverse portfolio (Not like the other two conditions they have a heavy weight on SPX). Now this study moves to analyze the images of these three cases. In Figure 6: Exclusion of Market Index, the difference between the two lines gets bigger as the risk gets bigger. Even before the max Sharpe ratio, with a standard deviation equal to 25 percent, the difference between the two lines is very large, implying that the return is unstable at a certain risk, even if the risk is small. Then we analyze the graph of no short selling, from the Sharpe ratio at the maximum risk rate to the left, the difference between the two lines is very small, and the minimum risk is close to the point of min variance. So, this is a good choice for risk-averse investors, who can choose between 13.5% and 19.5% risk under ideal assumptions. The most risk-averse investors can choose a risk of 13.5% to get a return of 7.79% to 13.67%; the relatively neutral risk-averse investors can choose to tend to 19.5%, then their Sharpe ratio is the largest, they can get a return of 24.85%. In the figure of Regulation T, from the risk ratio at the maximum of the Sharpe ratio to the left, the line of inefficiency tends to 0, and the difference between the two lines is large. The difference between the two lines is large. It is not until the min variance is almost reached that the variance becomes smaller, and investors who choose a risk of 13.0% will have a return range of 4.43% and 12.02%. Compared to the case of no short selling, the returns in the case of minimum risk are more volatile. In summary, in the index model, risk-averse investors should choose the no short selling constraint and choose a risk of 13.5% to 19.5% to optimize.

4.3 For Risk-tolerant Investor

Risk lovers in ideal conditions will undoubtedly choose "free problem" because it has the maximum Max Sharpe, its efficient frontier, and the capital allocation line overlap, the greater the risk, the greater the return. At a risk of 60%, the return has almost 100%. In this case, investors will borrow a lot of money to short, with high leverage, but also with big returns. But obviously, there are many restrictions, and the market will not allow investors to trade SPX, nor will it allow them to have high leverage. So, if one chooses between these four constraints and the "semi-ideal" condition, this study suggests that the investor chooses the second constraint, which is to limit the size of each stock. In this condition, the risk is greatly reduced, but the Sharpe ratio is still very significant. So, the investor can invest in the max Sharpe ratio corresponding to a risk of 22.5% and to the right of the axis, which means that the investor can choose a risk range of 22.5% to 60%. The more extreme the risk-loving investor will be, the more they will favor 60%, and they can get a maximum return of 76.19%, or they can lose 52.58%; then, the risk-loving investor who is neutral will favor 22.15% risk, and they can get a return of 34.15%, which is the best ratio of return risk.

5. Conclusion

This study set out the different investment strategies for investors who have different risk preferences by studying the efficient frontier, inefficient frontier, and the max Sharpe ratio using the single-index model. The most obvious finding to emerge from this study is that when risk-averse investors want to optimize investment portfolios, no short selling is the best constraint to reduce the risk. The risk-averse investors’ choice to risk will be between minimal risk and the return corresponding to max Sharpe. As for risk-seeking investors, ideally, there are no restrictions for them to optimize their portfolios. However, limiting the size of each stock to less than one is a good constraint for them, and they can choose what risk they decide to suffer corresponding to maximal and minimal return. These results can inform both kinds of investors of an expected range of their return respect with to risks they decide to take under certain constraints in the ideal situation.

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