Several Methods of Inventory Control

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Abstract. This paper would firstly introduce the Single-Period Inventory Model, a basic method of inventory control. By analyzing the Newsvendor Problem which is a simplified situation of storing inventories, this paper would discover a method using normal distribution analysis to solve the problem. Then, the paper would promote the adoption of this method into realistic inventory problems and discuss its utility. Secondly, the topic is expanded to Multi-Period Inventory System. This system focus on a more sophisticated inventory situation, and it could be basically categorized into two models: fixed-order quantity model, and fixed time period model. The paper would discuss the methods of ordering determination of the two models respectively.

Keywords: Single-period inventory system, Multi-period inventory system, Normal distribution, Coordination and overall planning.

1. Introduction

Production is the key of an industrial section, and the regulation of inventories is one of the most significant parts in production. In fact, inventory is kept to meet the variation in product demand and to allow flexibility in production scheduling.1 For centuries, people around the world have been looking for different methods to establish a model portraying the balance between products' demand and supply, and to design an inventory system which fits consumers' need and maximizes producers' revenue at the same time.

An eligible inventory system should monitor levels of inventory, determine what levels should be maintained, when the stocks should be replenished, and how large the orders should be. The basic purpose of inventory analysis should pay attention to when items should be ordered, and how large the order should be. Inventory systems could be classified into single-period systems and multi-period systems. At first, this paper would focus on single-period inventory system, of which the decision is a one-time purchasing decision where the purchase is designed to cover a fixed period of time and the item will not be reordered. Then the paper would turn to the more complicated multi-period inventory system, of which time and ordering times would be variables.

2. Single-period Inventory Systems

2.1 A Situation Faced by Newsvendor

Newsvendor is a simple job for children to gain their pocket money. A newsboy needs to buy in newspapers in weekend and sell them out on Monday morning. However, he have to determine how many papers to order in the weekend and then sell. This is an important decision which could affect the profit of the newsboy, because if he orders more newspapers then actually would sell, he will loss the cost of newspapers that will not have been sold; on the other hand, if he orders less newspapers to fill all his customers' demand, he will loss the revenue that would have been earned.

So, which algorithm could the newsboy use determine the number of newspapers to order to maximize his profit and to minimize the possibility of losing money?

2.2 A Specific Problem

2.2.1 Basic Details

First, assume that the newsboy needs to pay \$0.20 for each newspaper, and he would sell it for \$0.50. He has collected sales data over a few months and has found that on average each Monday 100 papers were sold with a standard deviation of 10 papers. (Here we assume that during this time the papers are at an overstock.) It's obvious that a normal distribution model should be adopted to portray this situation and deal with these statistics.

2.2.2 The Introduction of the Functions

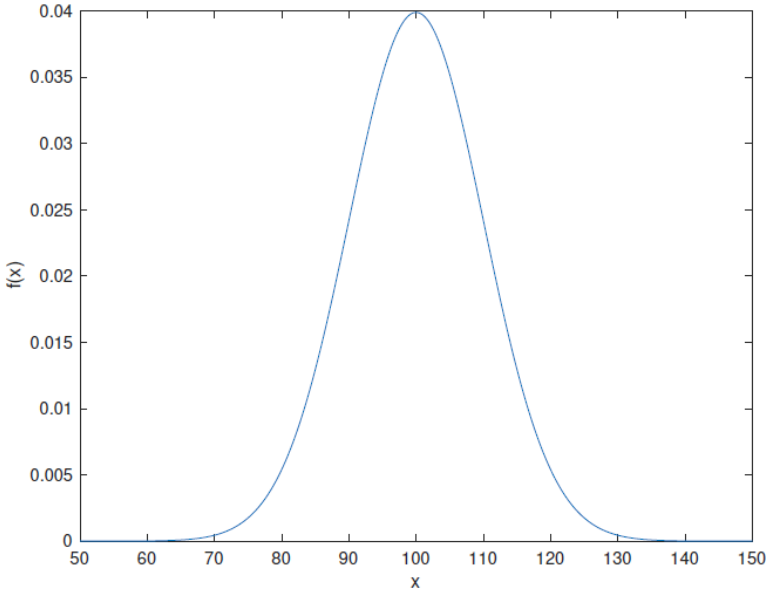

Figure 1. normal distribution of newspaper demand

The figure above shows the demand distribution model that could be used to portray this problem. Let this probability density function be $f(x)$.

Then,

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1}$$

According to the basic details mentioned before, $\tilde{d}\sim N(100,10^2)$ (where $\tilde{d}$ repre-sents the distribution of newspapers' demand).

So the equation could be

$$f(x)=\frac{1}{10\sqrt{2\pi}}e^{-\frac{(x|-100)^2}{200}}$$

Also, the distribution function could be defined as

$$F(x)=\int_{-\infty}^{x}f(t)\mathrm{d}t \tag{2}$$

This function is used to determine the accumulated probability of number of news-papers that is less than x.

Now, if the newsvendor wants to control the stocking out risk$\leqslant$50%, it is obvious that he need to order at least 100 newspapers.

However, if the newsvendor wants to control the stocking out risk$\leqslant$20%, the number of newspapers that he needs to order is more complicated and requires computing:

$$F(x)=\int_{-\infty}^{x}f(t)\mathrm{d}t=1-20\%$$

More generally, if the newsvendor wants to control the stocking out risk $\leqslant$ $(1-\mathrm{p})$, the equation below could be written down:

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = p \tag{3}$$

2.2.3 Standard Normal Distribution Model

The problem could be put in a more general model—the standard normal distribu-tion model, or $N(0, 1)$.[2]

In this model, $f(x) \doteq \varphi(x)$ and $F(x) \doteq \Phi(x)$, which means that

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{4}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\mathrm{d}t \tag{5}$$

This model could make calculation much easier due to the following lemma.

Lemma 1. $X \sim N(\mu, \sigma^2)$, $\frac{X-\mu}{\sigma} \sim N(0,1)$

Proof. The distribution function $F(x)$ for $\frac{X-\mu}{\sigma}$ is

$$F(x) = \mathbb{P}\{\frac{X-\mu}{\sigma} \le x\} = \mathbb{P}\{X \le \mu + \sigma x\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\mu+\sigma x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\mathrm{d}t$$

$$\text{Let } u = \frac{t-\mu}{\sigma},$$

$$\mathbb{P}\{\frac{X-\mu}{\sigma} \le x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\mathrm{d}t = \Phi(x)$$

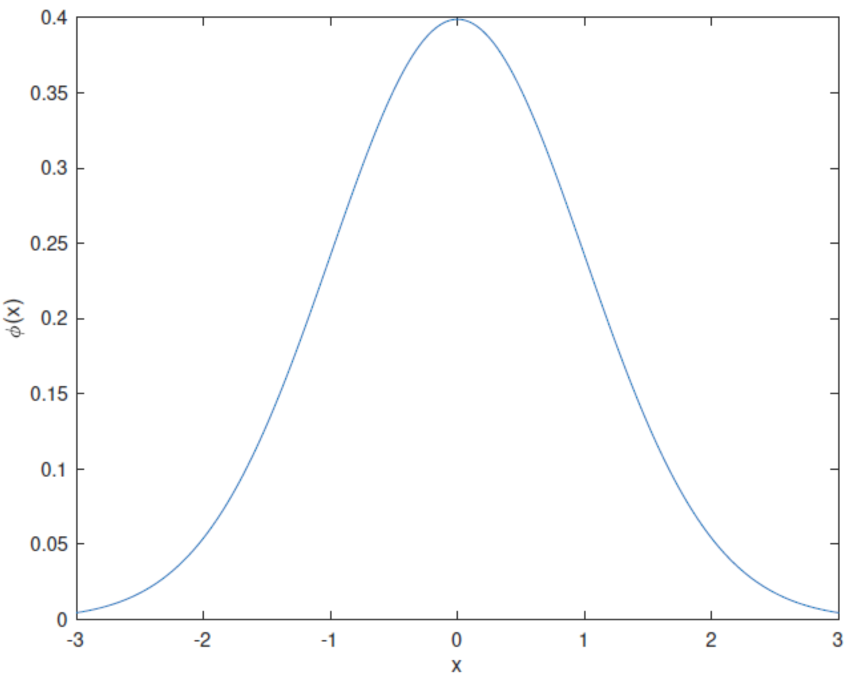

Figure 2. Standard Normal Distribution graph

Then for $X \sim N(\mu, \sigma^2)$, its distribution function could be written as

$$F(x) = \mathbb{P}\{X \le x\} = \mathbb{P}\{\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\} = \Phi(\frac{x-\mu}{\sigma})$$

While the newsvendor wants to control the stocking out risk $\leqslant$ $(1-p)$,

$$\mathbb{P}\{\tilde{d} \le x\} = \mathbb{P}\{\frac{\tilde{d}-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\} \ge p$$

According to the lemma, $\frac{\tilde{d}-\mu}{\sigma}$ could be considered as $N(0, 1)$. So, while the newsven- dor wants to control the stoσcking out risk $\le (1-p)$, the number of newspapers need to be ordered at least should be

$$x \ge \mu + \Phi^{-1}(p)\sigma \tag{6}$$

And in this problem,

$$x \ge 100 + 10\Phi^{-1}(p)$$

2.2.4 More Precise Calculation

The statistical picture deals with the risk of stocking out. However, it does not tell much about the profit or loss on the ordering because of the lack of costs in this model, which is just the ordering decision about. Marginal analysis is needed to determine the best probability of stocking out risk to control as well as the number of newspapers to order.

Recall that the newsvendor pays \$0.20 for each newspaper and sells each for \$0.50. Then, the marginal cost for over-ordering would be

$$C_o = 0.2$$

The marginal cost for under-ordering would be

$$C_u = 0.5 - 0.2 = 0.3$$

For one more object that is over-ordered, the expected marginal cost would be pC_o; conversely, for one less object that is under-ordered, the, the expected marginal cost would be $(1-p)C_u$. As under-ordering and over-ordering are mutually exclusive events, the expected marginal profit would be

$$(1-p)C_u - pC_o$$

While the order increases, the probability of the next unit that could be sold (or the stocking out probability) decreases, and the expected marginal profit would decrease. The best stocking level would happen when the expected marginal profit reaches 0, that is,

$$(1-p)C_u - pC_o = 0$$

Solving the equation above, the best probability would be

$$p_0 = \frac{C_u}{C_o + C_u} = \frac{0.3}{0.3 + 0.2} = 0.6$$

And the best number of newspapers to be ordered should be

$$x_0 = 100 + 10\Phi^{-1}(p_0) = 103$$

So while the newsvendor keeps 103 newspapers before Monday, he could expectedly earn the most profit on Monday.

2.3 Resolution Generalization

2.3.1 A Generalized Solution

This decision-making picture provides the best ordering decision. It also determines the best stocking probability level via a non-statistical approach (independent of the normality).

In a more general problem, the goal is to minimize the total cost which is composed of lost sales and surplus item costs for finished products. It could be depicted with the following variables: a marginal cost for over-ordering Co, a marginal cost for under-ordering Cu, the number x which is the number needed to be ordered, and a normal distribution d˜ with density f and probability F that could describe the demand distribution of the object.

The problem that need to be solved is to find out

$$\max \mathbb{E}[(C_o + C_u)\min\{\tilde{d}, x\}] - C_o x$$

By first-order condition, the marginal profit at an ordering decision x could be depicted by

$$\mathbb{P}\{\tilde{d} \geq x\}C_u - \mathbb{P}\{\tilde{d} \leq x\}C_o$$

Then, the best stocking probability p0 should be

$$p_0 = \frac{C_u}{C_o + C_u}$$

And the best number of this product to order x0 should be

$$\begin{aligned} x_0 &= F^{-1}\left(\frac{C_u}{C_o + C_u}\right) \\ &= \mu + \sigma\Phi^{-1}\left(\frac{C_u}{C_o + C_u}\right) \end{aligned}$$

2.3.2 Variable Analysis

We have already solved the answer for this genre of problem. Now, we would explore what would happen if the variables μ, σ, C_o, and Cu change, especially in particular situations.

μ: this variable represents the mean value of the number of products, which would only affect the number of products to order.

While other variables remain unchanged, if μ increases, the graph of normal distribution function moves rightward and the number of products to order should increase by the same value; conversely, with other variables remain unchanged, if μ decreases, the graph of normal distribution function moves leftward and the number of products to order should decrease by the same value.

σ: this variable represents the standard deviation of the number of products, which indicates the degree of dispersion of it.

While other variables remain unchanged, if σ is a relatively large number, the graph of normal distribution function would be flatter, the probability that the true $\tilde{d}$ drops near μ would be lower, and there would be more risks of over-ordering or under-ordering because of the increasing uncertainty. On the contrary, while other variables remain unchanged, if σ is a relatively small number, the graph of normal distribution function would be sharper, the probability that the true $\tilde{d}$ drops near σ would be higher, and the risk of over-ordering or under-ordering would be lower because of the increasing certainty.

C_o and C_u: the proportion between C_o and C_u could determine the best stocking probability and then determine the best number to order.

While $C_o \gg C_u$, $p_0 \to 0$, $x_0 \ll \mu$. This also conforms to economic meaning: while the cost of over-ordering is much greater than that of under-ordering, the risk of over-ordering becomes much greater too. To avoid the probable lost, it is better to order much less than the mean value.

While $C_o \ll C_u$, $p_0 \to 1$, $x_0 \gg \mu$. In economic meaning, while the cost of over- ordering is much less than that of under-ordering, the risk of over-ordering becomes much less too. To avoid the probable lost, it is better to order much more than the mean value. In this situation, no stocking out would be allowed.

2.4 An Example of Model Application

This single-period inventory model could be applied in numerous fields, so another example is given to show its convenience—the hotel reservation management.

The hotel reservation always suffers from last-minute cancellations, so most hotels would use an overbooking policy to reduce the potential loss. However, if the hotel is actually overbooked, it has to find a room in another hotel and pay the room-fee for the customer—that is another form of loss. To determine a reasonable, overbook- ing amount, the hotel could apply the Newsvendor Model.

Table 1. Statistical Information

No. of cancellation	Probability	Cumulative Probability
0	0.05	0.05
1	0.08	0.13
2	0.10	0.23
3	0.15	0.38
4	0.20	0.58
5	0.15	0.73
6	0.11	0.84
7	0.06	0.90
8	0.05	0.95
9	0.04	0.99
10	0.01	1.00

According to the table above, the number of last-minute cancellations could be approximately considered as a normal distribution with mean of 5 and standard deviation of 3.

More detailed information follows: the room rate for this hotel is $80, and for an- another hotel would be $200.

Now, the number of overbooking should be compared with the number of cancellations to determine the situation:

If the number of overbooking is greater than the number of cancellations, in other words, the hotel overestimates the amount of cancellations, then the marginal cost for each extra overbooking would be

$$C_o = 200$$

If the number of overbooking is less than the number of cancellations, in other words, the hotel underestimates the amount of cancellations, then the marginal cost for each extra cancellation would be

$$C_u = 80$$

According to the model, The best stock-safe (overbooking in this context) proba- bility would be

$$p_0 = \frac{C_u}{C_o + C_u} = \frac{80}{200 + 80} \simeq 0.286$$

Finally, the best overbooking number x_0 should be

$$x_0 = \mu + \sigma\Phi^{-1}(p_0) = 5 + 3\Phi^{-1}(0.286) =\simeq 3.302$$

Thus, in order to reduce the probable loss as much as possible, the hotel should let 3 rooms be overbooked.

2.5 Discussion

By the Newsvendor's Problem, a method to resolve single-period inventory models is shown and is generalized. The Newsvendor Model have numerous applications beside the two mentioned in this paper, such as dealing with the overbooking of airline flights, the ordering of fashion items, etc.. In fact, any genre of one-time purchasing decisions could be resolved by this model.

The single-period inventory system introduced in this paper is just a simplified model with relatively less variables. In more complex situations, holding component costs, tardiness penalties, and other factors should be considered to establish a more precise model. In addition, some new approaches to design discrete optimization algorithms are under developing.

3. Multiperiod Inventory Systems

Comparing with single-period inventory system, multi-period inventory systems are closer to the reality and more targeted, because more variables are considered in multiple periods of time. For an inventory system, there are two most important questions needed to be answered: when to order and how much to order. There are two basic genres: fixed-order quantity model and fixed-time period model.

3.1 Fixed-Order Quantity Model

3.1.1 Introduction

Fixed-order quantity model, also known as the Economic Order Quantity (EOQ) Model or Q-model, is an event-triggered model. The Q-model would initiate an order when the event of reaching a specific reorder level, R , occurs. Q-model's goal is to keep a balance between ordering cost and holding cost so that the inventories could always satisfy the estimated market demand and the cost is minimized at the same time.

3.1.2 Basic Assumptions

In order to establish a mathematical model, some basic assumptions are needed to be clarified:

First, the demand for the product must be continuous and constant with a constant demand rate;

Second, the lead time (time from ordering to receipt) must be constant; Third, the price per unit of product must be constant;

Fourth, holding cost must be proportional to average inventory and ordering cost must be constant;

Fifth, backorders must not be allowed, which means that all demands for the prod- uct must be satisfied.

To explain, notice that the system assumes a constant demand rate by which the inventory is reduced. When it reaches the reordering level, an order is placed. When products arrive, the inventory is replenished, and all at once the inventory is increased. However, new products cannot arrive just at the moment an order is placed as there is a certain amount of lead time during which we have to wait for the order.

3.1.3 Mathematical Analysis

It is obvious that the total annual cost is the sum of purchased cost, ordering cost, and holding cost. In other words,

$$f(Q) = DC + \frac{D}{Q}S + \frac{1}{2}QH \tag{7}$$

of which D represents annual demand, Q represents quantity to be ordered each time, C represents cost per unit, S represents ordering cost, and H represents annual holding cost per unit product of average inventory.

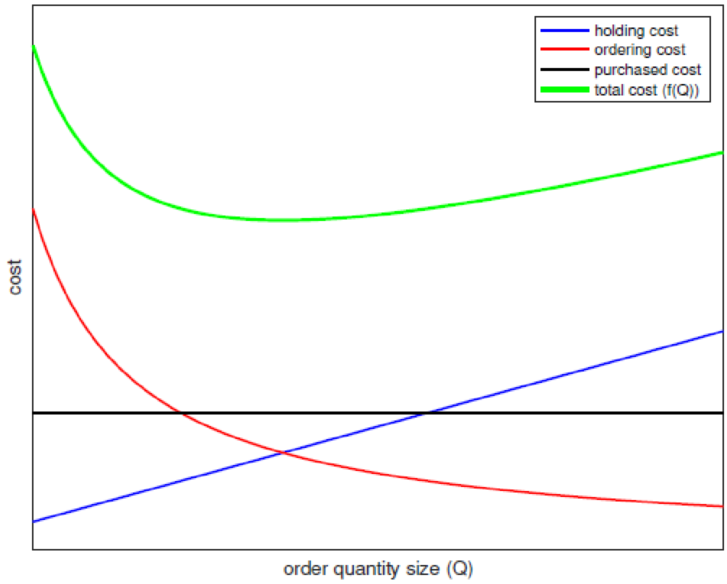

Figure 3. function graph

The major intention of Q-model is to determine R, the specific point, where a new order is needed, and Q, the number of products to order. As shown in the graph above, the total cost has its minimum, which means that $f(Q)$ has its minimum too, where

$$\frac{\mathrm{d}f(Q)}{\mathrm{d}Q}=0 \tag{8}$$

Then

$$\frac{\mathrm{d}f(Q)}{\mathrm{d}Q}=-\frac{DS}{Q^2}+\frac{1}{2}H=0$$

$$Q_0^2=\frac{2DS}{H}$$

$$Q_0=\sqrt{\frac{2DS}{H}}$$

and the reorder point R, which is actually the demand during lead time, as the inventory is daily checked, would be

$$R=\frac{1}{365}DL$$

3.1.4 A More General Situation

In the part above, an assumption is made that the demand for the product is con- stant at the end of a period, and the duration of each period are the same. However, the demand is not constant but various from day to day in practice. For example, when a new electronic device first appears in the market, the demand for it would rapidly increase because of producer's advertising behaviour and pre-sale; however, after this peak season, recession would come and the demand would decline.

Considering this change, a improved model is needed. In this new model, a safety- stock is needed to provide protection against stock-outs, and it is obvious that stock-out only emerges during lead time L.

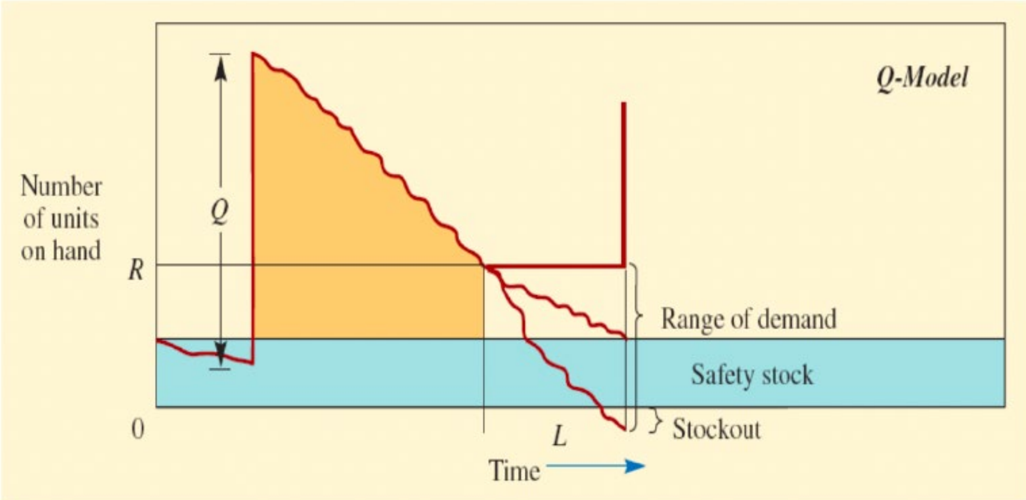

Figure 4. Safety-stock for uncertain demand

The ordering level Q could still be determined by the usual way by using average demand, but the reordering point R in this situation would be

R = Expected demand during L + safety-stock

Then the question comes:

How to define the safety-stock, so that the producer could control inventory level at a safe level in most situations?

In fact, the safety-stock is often defined such that the probability of stocking out is down to 5%, or in mathematics,

$$\mathbb{P}\{R \geq \text{demand in lead time}\} \geq 1 - 0.05$$

Let μ_L be the mean of demand during lead time and σ_L be the standard deviation of demand during lead time, such that the distribution of R becomes normal with $R \sim N(\mu_L, \sigma_L^2)$. Then, R could be computed as

$$R = \mu_L + \sigma_L \Phi^{-1}(0.95) \simeq \mu_L + 1.64\sigma_L$$

More generally, the reordering point R could be expressed as

$$R = \mu_L + \zeta\sigma_L$$

where $\zeta\sigma_L$ represents the safety-stock, then it could provide a safety-stock probability of

$$p_s = \Phi(\zeta) = \Phi(\frac{R - \mu_L}{\sigma_L})$$

Now, the problem is to compute μ_L and σ_L.

Case 1:

If the data of demand during lead time L is already provided, the formula could give answer directly.

For example, consider a case where the annual demand is $D = 1000$ units, the ordering quantity is $Q = 200$ units, the standard deviation of demand during lead time is $\sigma_L = 25$ units, the expected probability of not stocking out is $p = 0.95$, and lead time is $L = 15$ workdays. Assuming that there are 250 workdays, the average demand during L would be

$$\mu_L = \frac{1000}{250} \times 15 = 60$$

Then it is straightforward to determine R:

$$R = \mu_L + \sigma_L \Phi^{-1}(p) = 60 + 25 \times 1.64 = 101 \text{ (units)}$$

Case 2:

However, if only the daily demand data is provided, the calculation would be more complicated.

Now, assuming that for each day the daily demand is independent with average daily demand μd and standard deviation of daily demand σ_d.

Noticing

$$\mathrm{Var}(\tilde{d}_1 + \tilde{d}_2 + \ldots + \tilde{d}_L) = L\mathrm{Var}(\tilde{d}_i) \quad \Leftrightarrow \quad \sigma_L^2 = L\sigma_d^2$$

then

$$R = \mu_L + \zeta\sigma_L = \mu_d L + \zeta\sqrt{L\sigma_d^2}$$

Assuming that a n-day period demand data $d_i (i = 1, 2, \ldots, n)$ is provided, the mean and standard deviation would have a more precise result:

$$\mu_d = \frac{\sum_{i=1}^{n} d_i}{n}$$

$$\sigma_d = \sqrt{\frac{\sum_{i=1}^{n}(d_i - \bar{d})^2}{n-1}}$$

For example, the demand for a product that sells all year long is normally distributed with a mean of 60 units and standard deviation of 7 units. The supply source maintains a constant lead time of 6 days. The ordering cost is \$10 and annual holding cost is \$0.5 per unit. Assume that there are no stock-out costs and orders are filled as soon as the new products arrive.

In order to control the stock-out probability under 5%, consider the model where $\bar{d} = 60$, $D = 60 \times 365 = 21900$, $H = 0.5$, $S = 10$, $L = 6$. Then

$$Q = \sqrt{\frac{2DS}{H}} = 936 \text{ (units)}$$

$$R = \mu_d L + \zeta\sqrt{L\sigma_d^2} = 388 \text{ (units)}$$

3.1.5 Conclusion

The Q-model is introduced above. It favors more expensive items and the average inventory is lower. After over a century of development, it has become a significant tool for enterprises to maximize their profit during production. However, there is still space for improvement for some specific problems.

3.2 Fixed-Time Period Model

3.2.1 Introduction

Fixed-time period model, also known as Periodical System Model or P-model, is a time-triggered model just as its name suggests. Unlike Q-model, an order is placed only when it comes to the end of a predetermined time period, T . P-model would be used when a producer makes routine visits to customers and take orders for their complete line of products, or when a customer wants to combine orders to save on transportation costs. As a result, inventory is only counted at particular times.

3.2.2 General Resolution

For Q-model, stock-out only emerges during lead time. However, for P-model, the situation changes. It is possible that some large demands would make the stock down to zero during the whole period T + L. As a result, the safety-stock must also protect against stockout for the whole time period but not only for L.

In this model, the variable needed to be determined is the quantity for each order,

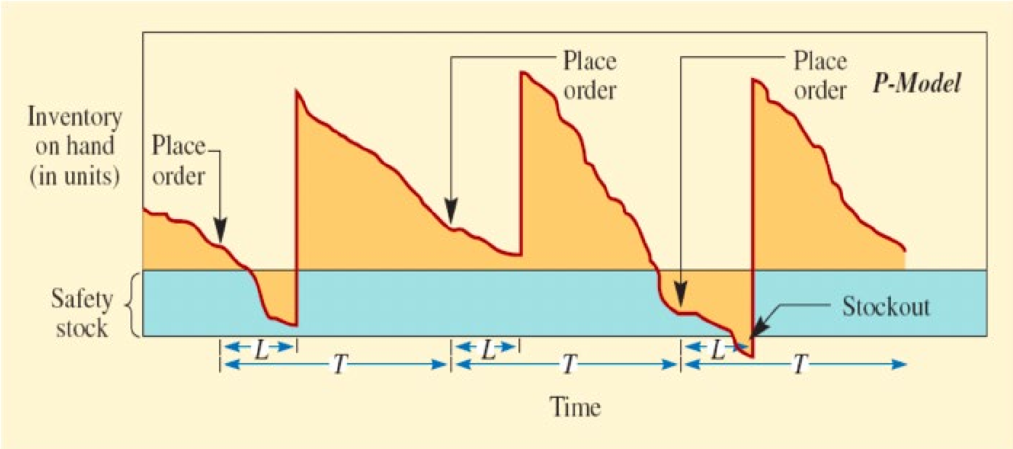

Figure 5. Fixed time period model

Let Q0 be the current inventory level, T be a whole period and L be the lead time during that period; let μT +L and σT +L be respectively the average demand and standard deviation during these T + L days, μd and σd be respectively the average daily demand and standard deviation of daily demand.

Referring to the calculation of Q-model above,

$$Q_0 + Q = \mu_{T+L} + \zeta\sigma_{T+L}$$

and

$$Q = \mu_{T+L} + \zeta\sigma_{T+L} - Q_0 = \mu_d(T+L) + \zeta\sqrt{(T+L)\sigma_d^2} - Q_0$$

As Q could be computed by the formula above, the problem becomes quite simple: just take a new order of Q products at the end of each period.

3.2.3 Conclusion

P-model generates ordering levels which vary from period to period depending on the products' usage rates. This inventory system requires a higher level of safety stock than a Q-model because a sudden surge of demand could lead to a stock-out, so it favors less expensive but more abundant items.

3.3 Comparaison Between The Two Systems

As the two systems each answers a question posed before, a great difference between Q-model and P-model is in the timing and quantities of the orders placed.

With Q-model, inventory is checked on a continual basis and the system is prepared to place orders multiple times per year on a random basis. This system is better at providing greater system responsiveness, because each order would be more flexible and controllable. However, a disadvantage is that it requires administrative processes which would increase workload.

P-model allows however more organized purchasing as inventory levels are checked in a fixed time interval. It could ensure that different items are bundled on a regular basis, which could provide an advantage comparing with Q-model of which different items may reach their reordering level at different moments and generating numerous orders at random time intervals.

4. Final Discussion

4.1 Conclusion

In this paper, two genres of models are introduced to determine the ordering level for inventory systems: Single-Period Inventory Model and Multi-Period Inventory Model. Based on probability theory, Single-Period Inventory Model provides a fundamental method of ordering level determination in a simple situation; then, Multi-Period Inventory Model extends and exploits this method by macro-planning in more complex situations with more variables.

4.2 Outlook and Improvement Suggestion

On the other hand, the models still have space for improvement. Considering the relationship of different products in one inventory system, both Q-model and P- model could be improved. Also, in order to evaluate the effectiveness of these models, the Inventory Turn could be considered. In addition, in reality, an enterprise often manages a huge combination of different merchandises. To cut holding costs, Pareto Principle needs to be considered too.

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