To what extent would an investment portfolio be affected by different variables in terms of Markowitz and Index model?

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Abstract. Portfolio Theory has been widely used in the securities market. Investors expect to maximize the return of the portfolio based on a given level of risk. By using naive diversification, investors can partly reduce the portfolio's risk by reducing the firm-specific influences. However, due to the macro-economic factors (inflation, interest rates, exchange rates, etc.), the risk cannot be eliminated entirely. Based on two popular models, the Index and Markowitz models, we chose seven stocks as one portfolio and set five constraints to simulate a real stock market. Even though results are very similar between Markowitz and Index model, the Markowitz model is more suitable than the index model in terms of circumstances we faced. And with any additional constraints added to the "free market," our portfolio return can only be negatively or non-affected. The purpose of this paper is to determine how the portfolio performance would be affected by different factors and how these two models would be used based on our comparative analysis of our portfolio in the index model and Markowitz model.

Keywords: Markowitz model, Index model, Portfolio Theory, constraints.

1. Introduction

Nobel laureate Harry Markowitz provided investors with a framework to optimize the risk and return of their portfolios of stocks and bonds. Evans and Archer first proved the relationship between risk and portfolio scale in the empirical study. They chose 470 stocks from S&P 500 and built 60 single securities,60 two securities portfolios, and 60 forty securities portfolios, respectively. Meanwhile, they calculated their standard deviation. The study showed for a portfolio of only two securities, and the standard deviation could be easily reduced by increasing another security. For a portfolio of 8 securities, the same number is 5. That is to say, when there are fewer securities in the portfolio, the standard deviation of the portfolio decreases rapidly with the expansion of the portfolio scale.

According to Markowitz, investors could virtually reduce their risks unique to individual securities by choosing stocks that do not move precisely together. This movement can provide a minimum risk for a given level of return [1]. Fischer and Lourie researched various portfolios and found that holding two stocks can reduce 40% of the non-systematic risk. When the portfolio consists of 128 stocks, the non-systematic risk can be decreased by 99%.

Due to the uncertainty of the market, Modern portfolio theory is an effective method for planning, introduced in Markowitz's landmark paper. It emphasizes that the risk of securities is not the only thing that investors focus on. Indeed, the contribution being generated by the securities to the entire portfolio variance is the other fundamental principle that investors focus on.[2]. In this situation, the emergence of portfolio theory provided a theoretical framework for the diversity of portfolios [3]. Researches have shown that a portfolio consisting of assets having highly positively correlated expected returns is, therefore, risky [4]. For most investors, maximizing expected return and, at the same time minimizing risk is their objective. In a portfolio context, risk can be reflected by the standard deviation (or volatility) of returns around their expected value [5].

Single-index modelling is widely used in, for example, econometric studies as a compromise between too restrictive parametric models and commercial analysis regarded as a standard. It is flexible but not suitable for the models without parametric. By such modelling, the statistical analysis usually focuses on estimating the index coefficients [6]. Index model can be an excellent method to simplify the estimation of covariance matrix problem and make the analysis of security expected return more applicable. The risk of a stock portfolio depends on the proportions of the individual stocks and their covariances [7]. Meanwhile, it allows measuring the risk components for particular securities and portfolios. Compared with the Markowitz model, the index model is simpler and needs fewer returns and variances.

Based on Markowitz, who is often called the father of modern portfolio theory[8], Markowitz's modern portfolio theory has a new selecting method of portfolios for investors who wish to form a portfolio with the highest expected return at a given level risk tolerance. Markowitz introduced a quantitative framework for portfolio selection and tried to optimize the deficiencies in the theory. His model is also well known as the mean-variance model. The mean-variance approach proposed by Markowitz was to solve the portfolio selection problem as it may represent the risk of the portfolios [9]. A decision-maker can select the optimal investing ratio to each security based on the sequent return rate. Following Markowitz, many researchers worked based on his model. They extended the portfolios' size and added a variety of assumptions, constraints, or objectives such as cardinality constraint, transaction cost, skewness, and kurtosis to his model to make it more realistic, which is the method we prepare to use as well [10]. And in this paper, we assume five different constraints to analyze the various performances that may happen to test the applications of these two models, respectively.

To achieve our target, we need to find the different factors that may happen under various constraints. And we choose some important elements in analyzing the trend of stocks. They are returning, standard deviation, sharp ratio. By comparing these differences, we found that Markowitz Model is more applicable for our portfolios. But in some situations, the index model also has its advantages. We also discovered that when additional constraints are added to the "free market," the return of our portfolio cannot be affected significantly.

The following paper will show the specific procedures of the comparison between the Index model and the Markowitz model. In particular, section 2 describes the data process and relevant analysis by using the statistical table. Section 3 shows the methodology we used to find the final result. Section 4 conducts the comparative analysis due to the performances expressed by different constraints and models. The last section presents our conclusions.

2. Data

We used Yahoo's database to gather historical daily total return data for six stocks, which belong in pairs to three different industry groups, one (S&P 500) equity index (a total of seven risky assets,) and a proxy for risk-free rate (1-month Fed Funds rate). The six companies are Microsoft, Oracle, JPMorgan Chase, Berkshire, Procter & Gamble, and Johnson & Johnson. The data we used spanned 20 years, from November 9, 2000, to November 10, 2020. To reduce non-Gaussian effects, we aggregated the daily data to the monthly observations, and based on those monthly observations, we calculated all proper optimization inputs for the full Markowitz Model (“MM”), alongside the Index Model (“IM”).

We plot the price of the (S&P 500) stock index and the price of a risk-free interest rate proxy 1-month Fed Funds rate).

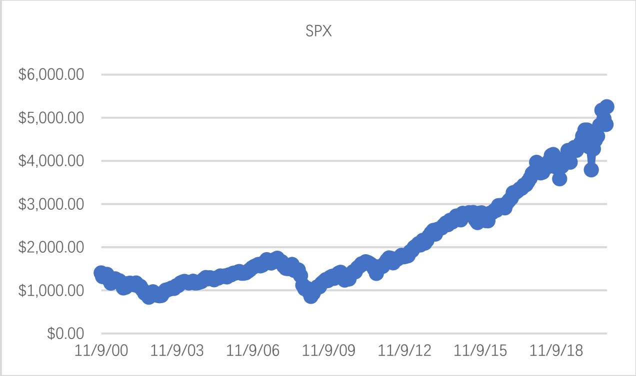

Fig. 1. yearly price volatility dynamics of SPX

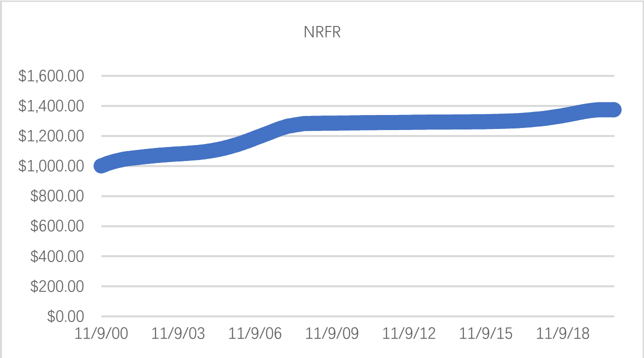

Fig. 2. yearly price volatility dynamics of 1-month Fed Funds rate

As can be seen from the bar chart of price changes (Fig. 1& Fig. 2), the prices of both have been on an upward trend in the past 20 years. The price of SPX has increased about fivefold, while the price of NRFR has increased about 1.4 times.

We selected two representative stocks from the six to plot a bar chart of their price movements. One is Microsoft from the computer industry, and the other is Berkshire from the financial industry.

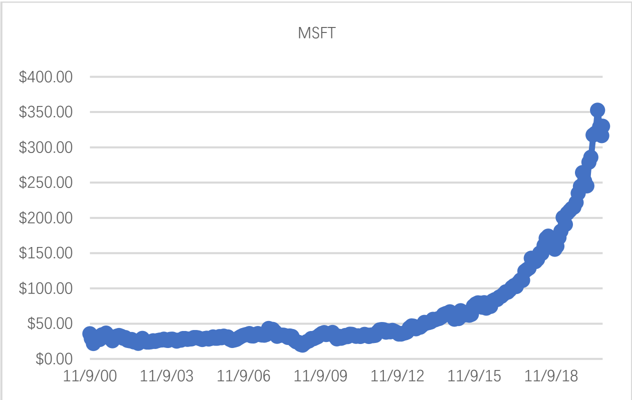

Fig. 3. yearly price volatility dynamics of Microsoft

It can be concluded from the bar chart (Fig. 3) that the stock price of Microsoft was in a stable state from 2000 to 2012. However, from 2012 to 2020, Microsoft's stock price began to rise rapidly. Its stock price is about 10 times higher than it was in 2012.

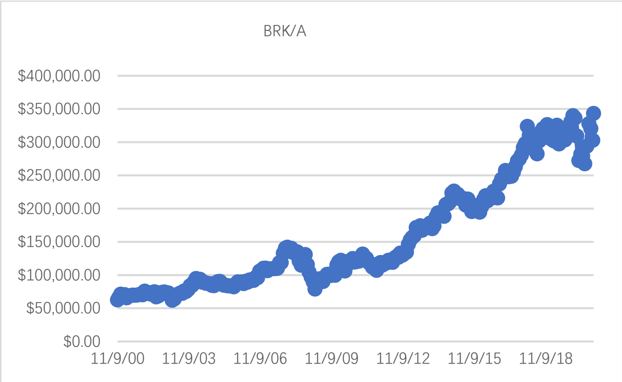

Fig. 4. yearly price volatility dynamics of Berkshire

From the bar chart (Fig. 4), it can be seen that the overall trend of Berkshire's stock price volatility is increasing, with a slight fluctuation in the middle. Berkshire's share price has been among the highest in the capital market for any single stock. By 2020 its price per share had risen to a staggering $343,000.

Table 1. Investment information of different assets

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
Annualized Average return	6.160%	12.971%	7.224%	10.776%	8.264%	9.207%	7.941%
Annualized StDev	15.096%	26.441%	29.223%	30.056%	16.334%	14.905%	15.381%
beta	1	1.066871	1.087515	1.38097	0.53665	0.39245	0.52762
Annualized alpha	0	0.063992	0.005253	0.0227	0.04958	0.06789	0.04691
Annualized residual StDev	0	0.209702	0.241754	0.21652	0.14183	0.13677	0.13158

According to our calculations (Table 1), Microsoft (MSFT) and JPMorgan (JPM) have relatively high average annualized returns of more than 10% each. Oracle (ORCL), Berkshire (BRKA), Procter & Gamble (PG), Johnson & Johnson (J&J), and SPX had relatively low annualized returns.

Microsoft (MSFT), Oracle (ORCL), and JPMorgan (JPM) have relatively higher annual standard deviations, which indicates that the stocks have relatively higher risks. Berkshire (BRKA), Procter & Gamble (PG), and Johnson & Johnson (J&J) are relatively low, which indicates that the stock risk of the stocks is limited. SPX has the lowest annual standard deviation, indicating the lowest risk.

The beta coefficient of Microsoft (MSFT), Oracle (ORCL), and JPMorgan (JPM) are more than one each, which indicates that the change range of earnings of these three stocks is larger than the change range of the market. The beta coefficient of Berkshire (BRKA), Procter & Gamble (PG), and Johnson & Johnson (J&J) is less than one each, which means the other three have moved less than the broader market. SPX normally corresponds to the stock market, with a beta equal to one.

All six stocks are greater than zero in terms of alpha, which means they are likely to be undervalued and likely to earn a higher than average expected return. SPX will earn an actual return equal to its expected return.

Table 2. Correlation coefficient of different assets.

Correlations	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
SPX	1.000	0.609	0.562	0.694	0.496	0.397	0.518
MSFT	0.609	1.000	0.416	0.437	0.184	0.099	0.211

ORCL	0.562	0.416	1.000	0.339	0.217	0.101	0.244
JPM	0.694	0.437	0.339	1.000	0.421	0.213	0.263
BRK/A	0.496	0.184	0.217	0.421	1.000	0.328	0.460
PG	0.397	0.099	0.101	0.213	0.328	1.000	0.518
JNJ	0.518	0.211	0.244	0.263	0.460	0.518	1.000

According to the correlation coefficient of different assets (Table 2), the six stocks we selected and one index fund positively correlated. The covariance is all less than zero, which indicates that our portfolio is relatively diversified.

Table 3. Descriptive statistics of variables.

Variable	Mean	Std.dev.	Min	Max
SPX	2102.51	1116.04	837.12	5255.77
MSFT	67.93	4815.37	19.40	352.56
ORCL	30.89	16.74	7.86	69.17
JPM	82.60	52.64	20.20	242.57
BRK/A	156204.48	82457.59	61700	343000
PG	94.86	46.10	30.36	241.28
JNJ	112.45	61.05	44.03	257.32

According to the following descriptive statistical analysis (Table 3) of the main variables, Berkshire has a wide range of stock prices. In contrast, Microsoft, JPMorgan Chase, Procter & Gamble, and Johnson & Johnson have relatively wide ranges. Oracle shares traded in a relatively narrow range

Berkshire and Microsoft have significantly higher average and standard deviation than the other four stocks, while Oracle has a lower average and standard deviation than the other five stocks.

3. Method

Our study aims to find the optimal portfolios of a combination of six stocks by processing with the Markowitz Model and Index Model. And using Markowitz Model and Index Model to stimulate some constraints.

3.1 Constraints

This additional optimization constraint is designed to simulate Regulation T., making the sum of the absolute value of weights lower than 2.

$$\sum_{i-1}^{7}|w_i| \leq 2 \tag{1}$$

The second constraint is designed to simulate some arbitrary "box" constraints on weights, which the client may provide.

$$|w_i| \leq 1, for\ \ \forall_i \tag{2}$$

The third constraint is designed to simulate the limitation of the short positions, which typically exist in the U.S.

$$w_i \geq 0, for\ \forall_i \tag{3}$$

And the last constraint is that we are setting the weight 1 to be zero.

$$w_1 = 0 \tag{4}$$

We consider the above constraints to establish different investment portfolios based on the Markowitz model and Index model.

3.2 Markowitz Model.

Markowitz Model is to make a portfolio of the placement of funds into a set, which gives the optimal returns with reasonable risk to investors. Markowitz's stock diversification model works on the returns of portfolio assets by combining that the returns have less than positive correlation and the returns to decrease portfolio variance and not reduce the returns. Moreover, Markowitz Model can defeat the weaknesses of stock diversification randomized.

$$R_i = (P_t - P_{t-1})/(P_{t-1}) \tag{5}$$

Where R_i is the return of stocks i, P_t is the price on the period t. P_{t-1} means the previous period.
For calculating the expected return of each stock:

$$E(R_i) = \frac{\sum_{j=1}^{n} R_{ij}}{N} \tag{6}$$

$E(R_i)$ is the expected return of i. R_{ij} is actual return of stock i in period j. N is the number of the observation period.

The risk of stock:

$$\sigma_i^2 = \sqrt{\frac{\sum_{i=1}^{n}[X_i - E(X_i)]^2}{n-1}} \tag{7}$$

σ_i is stock variant *i*, *X* is the return of stock *i*, n is the observation number of historical data.
For calculating the covariance:

$$\sigma_{RA,RB} = \sum_{i=1}^{n} \frac{\left[R_{A_i} - E(R_A) \cdot (R_{B_i} - E(R_B)\right]}{n} \tag{8}$$

In this formula, $\sigma_{RA,RB}$ is covariance returns between these two stocks. R_i is the return of the stocks by stating the stock A or B in condition *i*. And for more calculation of the correlation coefficient, we can use this formula:

$$R_{A,B} = \frac{\sum_{i=1}^{n}(RAi - RB) - N.E(RA).E(RB)}{\sqrt{[\sum_{i=1}^{n}(RAi^2 - n.E(RAi)^2][\sum_{i=1}^{n} RBi^2 - n.E(RBi)^2]}} \tag{9}$$

In this formula, $R_{A.B}$ is the correlation coefficient return of stock A and stock B. And Ri represents the future return of stocks in condition i. E(R) is the expected value of the stock return. N represents the number of observation periods. We use the formula to find the stocks portfolio by using Microsoft Excel's solver for more calculations. And the expected return portfolio can be calculated by the following formula:

$$E(Rp) = \sum_{i=1}^{n} Wi.E(Ri) \tag{10}$$

In this formula, E(Rp) is the expected return portfolio. And Wi is the weight of funds invested in stock i. E(Ri) is expected return stock i.

And the portfolio risk is calculated by this following formula:

$$\sigma_p^2 = \sum_{i=1}^{n} Wi \,.\, \sigma_i^2 + \sum_{i=j}^{n} W_i \,.W_j.\sigma_{ij} \tag{11}$$

σ_p^2 is the stock variant, σ_i^2 is stock return variant i. σ_{ij} represent the covariance among stocks. W is the weight of each stock.

3.3 Index Model.

Index Model developed by Sharpe with the theory of Index Model portfolio is a simplification of a theoretical model with far Markowitz Model portfolio by reducing the number of variables that need to be assessed. And the securities included in the Index Model are correlated and have a positive covariance because they all respond similarly to macroeconomic factors. Moreover, In Index Model, the covariance of each stock can be found by multiplying their betas and the market variance.
Mathematically the Index Model is expressed as:

$$r_{it} - r_f = \alpha_i + \beta_i\left(r_{mt} - r_f\right) + \epsilon_{it} \quad \epsilon_{it} \sim N(0, \sigma_i^2) \tag{12}$$

Where r_{it} is the return of stock i in period t. r_f is the risk-free rate, and r_{mt} represents the return to the market portfolio in period t. For presenting the stock's alpha and stock's beta, we use the symbols of σ and β, which mean the abnormal return and responsiveness to the market return, respectively.

3.4 Model Comparison

Our study aims to the optimal portfolio model of the data of 6 companies with a quantitative approach by using Microsoft Excel processed by calculating Markowitz Model and Index Model. In addition, we added 4 constraints on the calculations of the Markowitz Model and Index Model. And we focused on processing the data of the six companies on the two models and seeking the result of the optimal portfolio of all four constraints for considering both theories of the Index Model and Markowitz Model. Markowitz Model does not apply to the situation of a large total number of securities because it needs to calculate a huge amount of covariance data. And in this case, it will lead to large errors and inaccuracies in the results of calculations. In our study, however, we are only processing with 6 stocks, which means our calculation is considerably small, which can offset the Markowitz Model's weakness.

For comparing the Index Model and Markowitz Model. Index Model process a simpler calculation than Markowitz Model. Our study does not deal with a large amount of estimated data. Considering that there are only a small number of stocks in our portfolio, the simplification process of calculating the Index Model can lead to inaccurate results. And according to the theory of the Index Model, Index Model may conclude a wrong efficient frontier of the portfolio because the Index Model ignores the influence of non-market factors on the risk.

In our study, we process Markowitz Model and Index Model for calculating our data. Based on the Markowitz Model and Index Model theories, we concluded the optimal portfolios for the constraints of our six stocks. And depending on what we found and the Markowitz Model and Index Model theories, we found that Markowitz Model is more applicable for our portfolios. This is due to the relatively small amount of calculation we made can offset the weakness of the Markowitz Model.

4. RESULT ANALYSIS

This section will explain the results we computed for both the Markowitz model and index model under five different constraints, followed by a detailed comparison of the results. This section is mainly divided into two parts. Firstly, we would present all efficient risky portfolios under ten models and conduct comparative analysis based on the result we got under two theories: Markowitz and index model theory. Secondly, we compared and contrast each constraint's effect on portfolio return, volatility, and Sharpe ratio.

4.1 Comparison for Portfolio Theory

Table 4-14 exhibit the portfolio return, standard deviation, Sharpe ratio, and weights allocated on each stock in both efficient risky portfolio and mean-variance portfolio.

We discovered that both Markowitz and the index model produce exactly investment guidance on short/long positions based on our precise calculation and data processing. Under a free market, regulation T system, arbitrate limit policy, results show that investors are expected to short index fund and long the rest stocks since the performance of the S&P 500 have been dominated by other securities. In other words, some of the rest of the stocks are providing higher returns but with lower risk. Due to the same reason, when a short position is banned, the result shows that we should only invest in four stocks: Microsoft, Berkshire Hathaway, PG, and Johnson & Johnson, as their return and volatility dominate S&P500, Oracle, and JP Morgan.

Portfolio return and volatility in both models are also similar. Still, the return and risk of the optimal risk calculated by the Markowitz model are slightly greater than those calculated by the index model. Even though the weights they've imposed on each stock varied between 0 up to 17%, the final Sharpe ratio we computed only varied by 0.02-0.04. It suggests that both models produce reliable and effective forecast is predicting expected return and volatility and can be used when developing an investment strategy.

Nevertheless, in terms of the nature of the models, we believe that Markowitz is more reliable in our calculations. Even though the Markowitz Model's accuracy might be negatively affected in a situation where a large total number of securities needs to be taken into account, our current portfolio only considered seven stocks in total. Calculating the 20-year return rate, variance, and covariance drastically enhances the predictability and accountability of the results. At the same time, the covariance formula used in this model scientifically reveals that the key to risk diversification is to choose a portfolio composed of securities with a low correlation degree. In contrast, although the calculation process of the index model is less sophisticated than that of the Markowitz model, due to the small number of stocks in our portfolio, the interchange in the simplification process may result in an inaccurate expression of security risk. At the same time, the index model ignores the influence of non-market factors on the risk, which may lead to the wrong efficient frontier of the portfolio.

Table 4. Markowitz model in the free market

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ	Portfolio	Portfolio	Portfolio
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	Return	StDev	Sharpe
MaxSharpe	-0.7000	0.3655	0.0669	0.1072	0.3363	0.5998	0.2243	12.15%	14.14%	0.859
MinStDev	0.2784	0.0669	0.0155	-0.0941	0.2518	0.3437	0.1379	8.02%	11.49%	0.698

Table 5. Index model in the free market

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ	Portfolio	Portfolio	Portfolio
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	Return	StDev	Sharpe

MaxSharpe	-0.5386	0.2067	0.0128	0.0688	0.3501	0.5155	0.3849	10.89%	12.44%	0.876
MinStDev	0.2083	-0.0168	-0.0165	-0.0898	0.2545	0.3589	0.3015	7.78%	10.51%	0.740

Table 6. Markowitz model without index stock

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ	Portfolio	Portfolio	Portfolio
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	Return	StDev	Sharpe
MaxSharpe	0	0.2362	-0.0183	-0.0207	0.2462	0.4655	0.0910	9.75%	12.19%	0.800
MinStDev	0	0.1144	0.0516	-0.0440	0.2890	0.3948	0.1942	8.95%	11.68%	0.766

Table 7. Index model without index stock

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ	Portfolio	Portfolio	Portfolio
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	Return	StDev	Sharpe
MaxSharpe	0	0.1222	-0.0438	-0.0235	0.2502	0.4234	0.2716	9.14%	11.06%	0.826
MinStDev	0	0.0134	0.0065	-0.0548	0.2941	0.3938	0.3470	8.44%	10.64%	0.794

Table 8. Markowitz model when banning short position

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ	Portfolio	Portfolio	Portfolio
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	Return	StDev	Sharpe
MaxSharpe	0	0.2198	0	0	0.2303	0.4638	0.0861	9.71%	12.16%	0.798
MinStDev	0.1420	0.0663	0.0241	0	0.2327	0.3635	0.1714	8.54%	11.66%	0.732

Table 9. Index model when banning short position

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ	Portfolio	Portfolio	Portfolio
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	Return	StDev	Sharpe
MaxSharpe	0	0.1051	0	0	0.2324	0.4113	0.2512	9.07%	11.07%	0.819
MinStDev	0.0502	0	0	0	0.2642	0.3726	0.3130	8.41%	10.71%	0.785

Table 10. Markowitz model under arbitrage limit

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ	Portfolio	Portfolio	Portfolio

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	Return	StDev	Sharpe
MaxSharpe	-0.7000	0.3655	0.0669	0.1072	0.3363	0.5998	0.2243	12.15%	14.14%	0.859
MinStDev	0.2784	0.0669	0.0155	-0.0941	0.2518	0.3437	0.1379	8.02%	11.49%	0.698

Table 11. Index model under arbitrage limit

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ	Portfolio	Portfolio	Portfolio
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	Return	StDev	Sharpe
MaxSharpe	-0.5386	0.2067	0.0128	0.0688	0.3501	0.5155	0.3849	10.89%	12.44%	0.876
MinStDev	0.2083	-0.0168	-0.0165	-0.0898	0.2545	0.3589	0.3015	7.78%	10.51%	0.740

Table 12. Markowitz model under regulation T

SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ	portfolio	portfolio	portfolio
$\boldsymbol{w_1}$	w_2	w_3	w_4	w_5	w_6	w_7	Return	StDev	Sharpe
-0.19065	0.210045	0.040143	0.002406	0.292293	0.46646	0.179301	10.000%	12.151%	0.823

Table 13. Index model under regulation T

SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ	portfolio	portfolio	portfolio
$\boldsymbol{w_1}$	w_2	w_3	w_4	w_5	w_6	w_7	Return	StDev	Sharpe
-0.5	0.199746	0.009111	0.061886	0.34327	0.50865	0.377337	10.760%	12.290%	0.875

Table 14. Collective data

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
Annualized Average Return	6.16%	12.97%	7.22%	10.78%	8.26%	9.21%	7.94%
Annualized StDev	15.10%	26.44%	29.22%	30.06%	16.33%	14.90%	15.38%
Beta	1.0000	1.0669	1.0875	1.3810	0.5367	0.3924	0.5276
Annualized Alpha	0.0000	0.0640	0.0053	0.0227	0.0496	0.0679	0.0469
Annualized residual StDev	0.0000	0.2097	0.2418	0.2165	0.1418	0.1368	0.1316
Sharpe Ratio	0.4081	0.4906	0.2472	0.3585	0.5060	0.6177	0.5163

4.2 Comparison for Constraints

Taking "free market" as a controlled group, we aim to explore the influence of each constraint imposed on our existing risky portfolio in this section. Therefore, we summarized all the information in the tables below for better comparison.

4.2.1 With Index vs. Without index

By comparing the efficient risky portfolio that we conducted from both constraints, we discovered that the return in both Markowitz and Index models increases by about 2% by adding an index fund in the portfolio. In comparison, the standard deviation only increases by 1-2%. Since the return increases proportionally larger than the standard deviation, the Sharpe ratio has grown by adding S&P500 from 0.8 to 0.859 in the Index model and from 0.826 to 0.876 in the Markowitz model (shown as table 15). Therefore, we concluded that having an index fund in our portfolio generates a more positive effect on our portfolio as it brings more considerable revenue with limited risk.

Table 15. Efficient portfolio comparison between a portfolio with index and without index

With Index (Control group)	Markowitz	Index
Return	12.15%	10.89%
Standard deviation	14.14%	12.44%
Sharpe ratio	0.859	0.876
Without Index		
Return	9.75%	9.14%
Standard deviation	12.19%	11.06%
Sharpe ratio	0.8	0.826

Theoretically, investing in an index fund is considered an effective and cost-efficient passive investing strategy for investors to diversify portfolio risks. While the performance of each stock fluctuates over time, investing in a fund that holds 500 stocks equals the portfolio's performance to that of the index itself. Diversifying the portfolio among large quantities of companies ensures that the value of our portfolio is not overly correlated with any one of them. Hence, having an index fund in the portfolio is an effective strategy to boost portfolio return while stabilizing portfolio volatility.

4.2.2 Allowing Short Position vs. Banning Short Position

As shown in Table 16, under a situation where investors cannot take a short position, the return we can obtain from the portfolio is significantly lower than the free market. The Sharpe ratio in a shot-banning financial market is about 0.05 lower than the regular market. It decreases from 0.859 to 0.798 in the Markowitz model and falls from 0.876 to 0.781 in the Index model. The rate of return also drops by about 2.3% in total, while the standard deviation, only reduced by 1% by banning short position. Thus, we conclude from the model that restricting short positions reduces the investment opportunities, which leads to lower portfolio return and Sharpe ratio.

Table 16. Efficient portfolio comparison between portfolio allowing short position and banning short position

Banning short	Markowitz	Index
Return	9.71%	8.61%
Standard deviation	12.16%	11.02%
Sharpe ratio	0.798	0.781
Allowing short (control group)		
Return	12.15%	10.89%
Standard deviation	14.14%	12.44%
Sharpe ratio	0.859	0.876

This is because the market cannot use the benefits of a corrective force to prevent inferior stock bubbles from inflating. In reality, short-selling bans suggest that investors have to pay higher prices for stocks, which may potentially drive to stocks overvaluation and a less efficient and less sustainable in the long-term, as investors under such situation can only long best-performing stocks instead of shorting the stock to correct its prices.

Therefore, to maximize risk-adjusted portfolio return, rational investors would only allocate their funds on four stocks that exhibit the highest shape ratios, refusing to invest in S&P500, Oracle, and JP Morgan (as indicated in table #). And the weights are also distributed according to their respective Sharpe ratio, meaning the weights would be highest in PG, followed by J&J, Berkshire Hathaway, and Microsoft (refer to Table 11).

4.2.3 With arbitrage limit vs. Without arbitrage limit

By comparing the efficient risky portfolio, we produced from both constraints. We discovered that the arbitrage limit has no impact on our existing portfolio. In a Markowitz model, the return for both conditions is 12.15%, and the volatility is 14.14%. And in an index model, the return and standard deviation are also the same for both portfolios. This is primarily because our original efficient point is already within the arbitrage constraint. The absolute weights allocated on each stock are already lower than one, so the additional arbitrage constraint would not influence our efficient portfolio.

Table 17. Efficient portfolio comparison between a portfolio with arbitrage limit and without arbitrage limit

With Arbitrage Limit	Markowitz	Index
Return	12.15%	10.89%
Standard deviation	14.14%	12.44%
Sharpe ratio	0.859	0.876
Without Arbitrage Limit (control group)		
Return	12.15%	10.89%
Standard deviation	14.14%	12.44%
Sharpe ratio	0.859	0.876

The result suggests that in many cases, even though there are policies to limit arbitrage in the financial market, in a situation like this, its impact on our portfolio return and portfolio volatility is limited, as the arbitrage opportunity of investing heavily in one individual stock is uncommon.

As a result, arbitrary "box" constraints on weights (which set the absolute weights to be less than 1) seem to be a generous constraint. If the arbitrary restrictions on weights continuously decrease from 1 to 0.5, our efficient risky portfolio might be affected. However, at the current stage, the additional arbitrage limit has not impacted our portfolio. The weights allocated in each stock, the return that we can obtain, and the risk of the efficient portfolio are precisely the same.

4.2.4 With Regulation T vs. Without regulation T

According to Table 18, we discovered that with the Federal Reserve Board regulation T, our portfolio risk-adjusted return had been negatively affected. The rate of return decreases by about 2% in the Markowitz model and 0.1% in the index model. The volatility also reduces by about 2% in Markowitz and 0.1% in the index model since the amount people can invest in stock has an upper boundary. As a result, Sharpe ratios also diminish.

Table 18. Efficient portfolio comparison between a portfolio with regulation T and without regulation T

With regulation T	Markowitz	Index
Return	10%	10.76%
Standard deviation	12.151%	12.29%
Sharpe ratio	0.823	0.875
Without regulation T (control group)		

Return	12.15%	10.89%
Standard deviation	14.14%	12.44%
Sharpe ratio	0.859	0.876

Nevertheless, the investment strategy under Regulation T is very similar to the control group, which is short heavily on stock index S&P500 and long PG and Berkshire Hathaway. The regulation's influence on our existing portfolio is insignificant since our total absolute weight only exceeds the constraint by a small number. Therefore, we may conclude that our portfolio would only be slightly affected having Federal Reserve Board Regulation T implemented.

In summary, we discovered that with any additional constraints added to the "free market," our portfolio return can only be negatively or non-affected. If the index stock is not taken into account, short shelling is banning, and regulation T implemented, the risk-adjusted return all diminishes. When arbitrary "box" constraints on weights are only 1, the portfolio is unaffected.

5. Conclusion

Portfolio theory has been widely used in the security market. Investors want to maximize the return of their portfolios for a given level of risk. By using pure diversification, investors can partially mitigate portfolio risk by reducing company-specific influences. Based on the index model and Markowitz model, this paper selects 7 stocks as an investment portfolio, sets 5 constraints to simulate the real stock market, and compares the two models and 5 constraints.

In our study, we process Markowitz Model and Index Model for calculating our data. Based on the Markowitz Model and Index Model theories, we concluded the optimal portfolios for the constraints of our six stocks. And depending on what we found and the Markowitz Model and Index Model theories, we found that Markowitz Model is more applicable for our portfolios. This is due to the relatively small amount of calculation we made can offset the weakness of the Markowitz Model.

Through analyzing the results, we got for our models. We discovered that multiple variables affect portfolio structure. We discovered that the result they produced for portfolio return, volatility, and Sharpe ratio is very similar for the comparisons of the two models. However, due to their respective nature and principle, we believe the Markowitz model is more applicable in our calculations since we only examined seven stocks in total. Between different constraints, we discovered that with any additional constraints added to the "free market," our portfolio return can only be negatively or non-affected. When the index stock is not taken into account, portfolio return, standard deviation, and Sharpe ratio decrease due to the diminishing diversifying effect. Banning short shelling also cuts the investment opportunities and leads to a lower return since it reduces the market corrective force to help prevent inferior stock bubbles from inflating. Investors can only long-best perform stocks rather than short selling inferior stocks. The impact of regulation T on our portfolio is insignificant, and the arbitrage limit does not affect our result since the original portfolio composition is within the constraint.

This paper has two main defects. First, we did not choose enough samples to test whether more principles could be discovered in terms of complex calculations. Second, the sample securities all display an above-average performance which all beat the benchmark and may underrepresent the market. Due to these factors, the samples we choose may have selective bias, and our constraints may not represent the actual situation in the market. In the future, we could combine more stocks into a portfolio and choose the samples representing the market's representativeness. Meanwhile, we will assume more constraints to detect the variances in the portfolio.

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