

Portfolio Establishment for Asset Allocation based on the Empirical Study

Qinxin Wang^{1,†}, Yuxin Tang^{2,†}, Yanyan Jin^{3,*}, Yutong Lai^{4,*},
Runyin Wang^{5,†}, Yihua He^{6,†}

¹Department of Accountancy, Central University of Finance and Economics, Beijing, China

²Candidate for Bachelor of Science Degree in Mathematics,
Northeastern University, Boston, MA US

³Candidate for Bachelor of Science in Mathematics and Business Administration, Northeastern University, Boston, MA US

⁴Professional Accountancy, The Chinese University of Hong Kong, Shenzhen, China

⁵Finance, Henan university of Economics and Law, Zhengzhou, China

⁶Finance, University of International Business and Economy, Beijing, China

*Corresponding author: qinxinw727@gmail.com

tommy19991230@outlook.com

jin.yan@northeastern.edu

118020223@link.cuhk.edu.cn

1404454096@qq.com

18610032351@163.com

†These authors contributed equally.

Abstract. The purpose of this study is to use Markowitz model and Single-factor index model to determine the optimal portfolio of S&P 500 and 6 stocks from 3 industries under 5 different constraints. The research is looked at stock prices on a daily basis from January 2000 to December 2020 and all data were obtained from secondary sources or non-participatory observation methods. Then this paper also considers the sensitive analysis to investigate the different impacts of factors. The sensitivity analysis tool used in this study was the Solver Table. Through calculation, the optimal portfolio ratio, minimum risk point, maximum Sharpe ratio, effective frontier, minimum return frontier and maximum variance frontier for each constraint under different models are obtained. In this process, it has been found that Single-factor Index model is simpler and more practical than Markowitz model. This paper provides inspiration for individual investors and financial institutions that Index model is more convenient to use in real life and likewise help them to make investment decisions.

Keywords: Optimal portfolio; Markowitz model; Single-factor Index model.

1. Introduction

When investing money in a portfolio composing of several stocks, there will be two based sources of risks, which are macroeconomic risks and firm-specific risks [1]. Macroeconomic influences refer to risks that comes from the general economy's conditions, such as the business cycle, inflation, and interest rates. Firm-specific influences might be risks that can be derived from investors' advance in research and development and also their attitudes on risk aversion. In order to control the risks of a portfolio, it is necessary to invest money into several assets to make the portfolio diversified. By allocating money into different assets that this study deal with according to specific weights, portfolio optimization could be achieved [4, 5]. If this article can find the optimal portfolio point on the

expected return to standard deviation graph, there will be the highest sharp ratio; in other words, a maximum return can be gained by increasing one unit of risks.

There are two main methods to create a portfolio optimization. The first method is called the Markowitz Model [2] and the second method is called the Index Model [3]. When using the Markowitz Model, there will be a huge number of estimates to fill the covariance matrix. And there are also no guidelines for finding useful estimates to construct the efficient frontier. However, the Index Model simplifies covariance matrix estimation and promotes security risk premiums analysis by explicitly categorizing into only systematic and firm-specific elements. In addition, sometimes investors would apply constraints while creating the investing portfolio. Therefore, this paper also analyzed the optimal portfolios under different constraints for these two methods.

For nearly half a century, there have been so many people considered efficient investment management. Different scholars build different portfolios which are based on their special viewpoint and model to fit the change in the market. Brinson et al., [6] used a framework based on a passive, benchmark portfolio and weighted by long-term allocations to evaluate the investment policy, marketing timing, and security selection. Fama and French [7] proposed a new definite return of bonds by using the factors of two bond-market factors, related to maturity and default risks, and Fama and French's five factors model is a significant way to evaluate the value of the portfolio. Asness [8] define quality as a characteristic that means that investors are willing to pay higher for, and they define quality-minus-junk factor and long high-quality stocks and shorts low-quality stocks. Lev and Srivastava [9] tried to overcome the systematic problem of accounting deficiencies and the significant slowed down reshuffling of value and identifying the portfolio still generate gains in recent years.

The Markowitz model was first published by Markowitz [10] in his article Foundations of portfolio theory, and he used the return and standard deviation of stocks to build portfolios. There are several discussions based on this theory. Pogue [11] consider several generally neglected issues such as brokerage charges and price effects associated with large volume transactions. This paper includes taxes on portfolio capital gains and short sales and liabilities as well. Soleimani et al. [12] proposed to include three limitations to make the Markowitz model more practical, which are Minimum transaction lots, cardinality constraints, and market capitalization.

The Single Index Model is important in portfolio analysis and has a great influence in practice. Sharpe [13] first put forward this method in 1963, he found the possibility of low-cost analysis and at the same time sacrificed relatively small information of stocks. And this model is an initial practical application of the Markowitz technique. Hristache et al. [14] further develop the index model by using improving the model to do statistical analysis and do the iterative improvements of the original ADE. Bilbao et al. [15] proposed to set portfolio based on index model and include imprecise investor's aspirations concerning asset's proportion of both high-risk and low-risk assets.

In this paper, the single index model and Markowitz model are used to construct the optimal solution portfolio for six selected stocks in the US stock market. The six selected stocks are MSFT, ORCL, JPM, PRK/A, PG and JNJ from three industries which are technology, financial service and health care. Furthermore, this study considers that countries' regulation of stock position, investor's preference may impose constraint on optimal portfolio solutions. Thus, five constraints were added to the weight allocation process. Later, weight solutions obtained from the five constraint were compared within each model. To compare the solutions between the two models, efficient frontier, minimal risk frontier and minimal risk point was constructed. In this article, judging from both the single index model and Markowitz model, constraint 2 and constraint 3 offer the solutions with the highest return. The calculation procedure of the single index model was considered much simpler than Markowitz model's.

The remainder of the paper is organized as follows: Section 2 briefly describes the company of each stocks and the industries they are situated in. In Section 3, the data collecting process was covered which includes the sources and description of the data, the way to construct the five constraints and the way to come up with beta for each stock. The meaning and definition of the five constraint and beta was also mentioned. Session 4 firstly introduces the single index model and then

shows the results and comparison between the five constraint's value. Session 5 introduces the Markowitz model and exhibit the results and comparison between the five constraint's value. Session 6 sums up all the results this study gained. The last section presents the conclusion.

2. Background Description

This chapter will give a brief introduction to all the enterprises whose stock were included in the portfolio. All of them are multi-national companies with a highly advantageous position in their corresponding fields, providing a reliable and stable return.

2.1 Microsoft

Microsoft Corporation is an American multinational technology company with headquarters in Redmond, Washington. It develops, manufactures, licenses, supports, and sells computer software, consumer electronics, personal computers, and related services. Its best known software products are the Microsoft Windows line of operating systems, the Microsoft Office suite, and the Internet Explorer and Edge web browsers. It is considered one of the Big Five companies in the U.S. information technology industry, along with Google, Apple, Amazon, and Facebook.

2.2 Oracle

Oracle Corporation is an American multinational computer technology corporation headquartered in Austin, Texas. The company specialize in database software and technology, cloud engineered systems, and enterprise software products—particularly its own brands of database management systems. In 2019, Oracle was the second-largest software company by revenue and market capitalization, just lower than first company, that is, MSFT.

2.3 JP Morgan&Co.

JPMorgan Chase & Co. is an American multinational investment bank and financial services holding company headquartered in New York City. JPMorgan Chase is ranked by S&P Global as the largest bank in the United States and the seventh largest bank in the world by total assets, with total assets of US\$3.213 trillion. It is also the world's most valuable bank by market capitalization.

2.4 Berkshire Hathaway

Berkshire Hathaway is an American multinational conglomerate holding company headquartered in Omaha, Nebraska, United States. The company wholly owns Duracell, Dairy Queen, BNSF and seven more companies, and also owns a significant minority holdings in public companies like Kraft Heinz Company, American Express, The Coca-Cola Company, Bank of America, and Apple.

The company is known for its control and leadership by the legendary Warren Buffett himself, who serves as chairman and chief executive, and by Charlie Munger, one of the company's vice-chairmen. Buffett recently focus more on buying whole companies.

2.5 Procter & Gamble

The Procter & Gamble Company (P&G) is an American multinational consumer goods corporation headquartered in Cincinnati, Ohio, founded in 1837 by William Procter and James Gamble. It specializes in a wide range of consumer health, and personal care and hygiene products.

For the fiscal year 2018, Procter & Gamble reported earnings of US\$9.750 billion, with an annual revenue of US\$66.832 billion, an increase of 2.7% over the previous fiscal cycle. Procter & Gamble ranked No. 42 on the 2018 Fortune 500 list of the largest United States corporations by total revenue.

2.6 Jonson & Jonson

Johnson & Johnson (J&J) is an American multinational corporation founded in 1886. Headquartered in New Brunswick, New Jersey, the company develops medical devices,

pharmaceuticals, and consumer packaged goods. Its common stock is a component of the Dow Jones Industrial Average and the company is ranked No. 37 on the 2018 Fortune 500 list of the largest United States corporations by total revenue. Johnson & Johnson is one of the world's most valuable companies and is one of only two U.S.-based companies that has a prime credit rating of AAA, higher than that of the United States government.

3. Data Collecting

The data dealt with is the 20 years of historical daily total return data for six stocks from Yahoo Finance. Those six stocks belong to in pairs of three different industry groups, one (S&P 500) equity index (a total of seven risky assets) and a proxy for risk-free rate (1-month Fed Fund rate). Specifically, the dates are from 11/09/2000 to 11/10/2020. This project considers the S&P 500 as the market benchmark for capitalization.

Here are the two statistical charts for S&P 500 prices that have been plotted by using RStudio. The first chart describes the prices of S&P 500 respecting to each day in the data set, so a general idea about the volatility of the market benchmark will be got. And the second chart demonstrates the daily log return of S&P 500 for those 20 years.

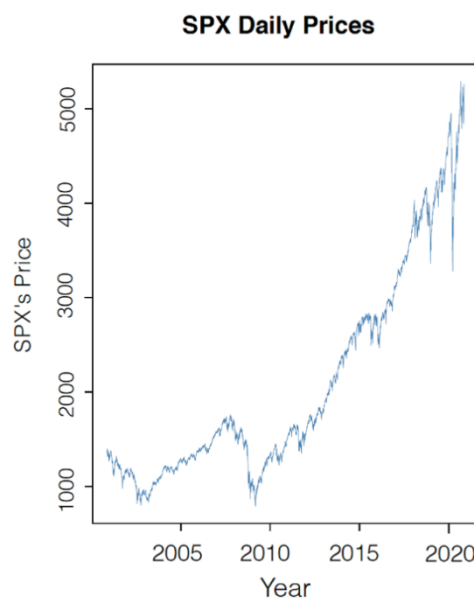


Fig. 1 S&P 500 daily prices volatility for these 20 years

From the figure 1, it is obvious that there was a steep decreasing in the S&P 500 prices around 2008 and 2020. The huge decreasing in prices around 2008 might be due to the great financial recession at that time. And the other huge decreasing in prices around 2020 might be due to the coronavirus pandemic. However, if looking at this chart from a broader perspective, it could be found that there is a continuous increasing in the S&P 500 prices starting from 2008 after the huge decreasing.

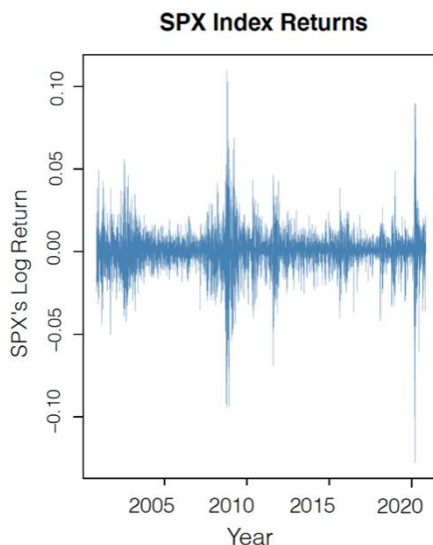


Fig. 2 S&P 500 Daily log returns for these 20 years

Figure 2 shows that the S&P 500 index return for those 20 years almost follows a pattern of fluctuation around 0. However, there were evident price jumps in 2008 and 2020, which corresponds to those steep decreasing in prices in the previous daily price chart.

Then this study created a statistical summary table for all the variables dealt with in the project. By doing such a statistical table, there will be a clearer perspective about our stocks, such as which stock would have the highest mean, or which stock would have the highest standard deviation. In this case, when doing the optimization by using these two different models, a deeper comprehension about the weights that has been gained from the calculations will be achieved.

Table 1. Descriptive statistics for variables.

Variables	Mean	Std. dev.	Min	Max
SPX	2092.2000	1101.3710	788.4000	5293.0000
MSFT	66.9000	67.2705	18.2000	362.1300
ORCL	30.8150	16.6407	7.3210	71.4900
JPM	82.2800	52.4077	16.7400	245.5100
BRK/A	155622.0	81803.0300	61200.0	344970.0
PG	94.6500	45.1252	28.6300	247.1800
JNJ	112.2700	60.6381	41.9100	258.0200
FEDL01	1.5370	1.6924	0.0400	6.6700

Table 1 indicates that for those 6 stocks this study worked with, the stock price of BRK/A over these 20 years has the highest mean value. And BRK/A also has the highest standard deviation among all the stocks this study had.

This project calculated the proper optimization inputs for both the full Markowitz Model (“MM”) and the Index Model (“IM”). The Markowitz Model is a portfolio optimization model which shows investors how to reduce their risks by choosing securities with different moving directions. For the Index Model, this article focused on the single-index model, which measures both the risk and return of a stock, assuming that there is only 1 macroeconomic factor that causes the systematic risk affecting stocks’ returns. In order to reduce non-Gaussian effect, the data has been aggregated to monthly observations. For each scenario that has been calculated, the target is to find the regions of permissible portfolios for five cases of additional constraints; such permissible portfolios are efficient frontier, minimal risk portfolio, optimal portfolio, and minimal portfolios frontier. Then, this paper visualized the gained results by drawing line charts in excel and compared the results with the same

constraint under different models. Based on the analysis above, this paper explained and summarized the differences that have been found during the research.

The first constraint that has been used is designed to simulate the Regulation T by FINRA. The formula used here is $\sum_{i=1}^7 |w_i| \leq 2$, where w_i denotes the weights of each assets in the portfolio. Regulation T is a collection of provisions that govern investors' cash accounts and the amount of credit, which both brokerage firms and dealers may extend to customers for the purchase of securities. According to Regulation T, an investor may borrow up to 50% of the purchase price of securities that can be bought using a loan from a broker or dealer. The remaining 50% of the price must be funded with cash. The rules of FINRA required customers' accounts to maintain margin requirements. Failure to do so would cause the firm to liquidate customers' securities in order to bring their accounts' equity back to the required level [16].

The second constraint is designed to simulate some arbitrary "box" constraints on weights, which may be provided by the client. It seems that investors have a tendency of not making too much transaction on one single stock. The formula for applying this constraint is $|w_i| \leq 1, for \forall_i$.

The third constrain is free, a calculation with no constraints. Therefore, there is no need to use any formulas when having this constraint in the portfolio. By applying this constraint, there will be a more basic form to illustrate how the area of permissible portfolios in general and the efficient frontier would be look like if there are no constraints.

The fourth constraint is designed to simulate the typical limitations in the U.S. mutual fund industry. The corresponding that has been applied in this model is $w_i \geq 0, for \forall_i$. According to the limitations in the U.S. mutual fund industry, any open-ended mutual funds are not allowed to be in a short position, expect the short-term credits are used for clearance and transactions [17].

The last constraint is designed to test whether the inclusion of the broad index in the portfolio would cause a positive or negative effect. To achieve the goal, this study set the weight of the first stock to be 0. And the formula could be written as $w_i = 0$.

Then, the beta will be explained. By definition, beta is a measure of the volatility of a portfolio compared to the market as a whole. By calculating the beta of an individual asset, this study gained an approximation on how much risk this asset would add to the diversified portfolio [18]. In the Index model, it is beta which describes the relationship between the systematic risk and expected returns for assets. One thing to notice is that for beta to provide any useful insight, the market benchmark should be related to the stock. Beta is calculated by dividing the product of the covariance of the security's returns and the market's returns by the variance of the market's returns over a specified period. However, beta calculation cannot detect any unsystematic risk. In addition, since beta is calculated based on historical data, it becomes less meaningful for investors when they want to predict a stock's future movements.

There are four main types of beta values. Firstly, beta can be equal to 0. In this case, it could be concluded that the stock's price activity is strongly correlated with the market. And adding such an asset with $\beta = 1$ doesn't add neither risks nor excess returns to the portfolio. The second type would be beta with less than 1, which indicates that the security is theoretically less volatile than the market. Adding such an asset with $\beta < 1$ will decrease the portfolio's risk. An example for such an asset can be utility stocks because they tend to move more slowly than market averages. Instead of being less than 1, beta can also be greater than 1. Such a value indicates that the security is theoretically more volatile than the market, adding such an asset with $\beta > 1$ will increase the portfolio's risk. Technology stocks and small cap stocks are good examples. Finally, beta can be less than 0. In this case, it is clear that the security should be inversely correlated with the market benchmark.

Looking at the data that has been dealt with in this project, it could be easily noticed that Microsoft, Oracle, and JP. Morgan Chase have a beta greater than 1, so they are more volatile than the market. Therefore, adding these assets to the portfolio will increase the total risk. However, Berkshire Hathaway, Procter and Gamble, Johnson & Johnson have a beta less than 1, which means that the volatility of these three assets is less than that of the market. As a result, adding these assets to the portfolio will decrease the total risk.

This project applied the Markowitz Model and the Index Model to the portfolio respectively and calculated the corresponding optimal portfolios. It also tried to find the permissible portfolios under each five constraint for these two models in this paper and then, compare the results gained from it.

4. Index Model

Single index model is one of the asset pricing models that focused on the return of a stock. The component β refers to the market responsiveness of a portfolio's return with the market's return. Alpha measures the abnormal return of the stock or portfolio differentiated from the market. The researcher used the six stocks and by imputing their individual return and standard deviation came up with their portfolio return and standard deviation given certain level of weights. The optimal portfolio solutions of different constraint were developed by seeking for the highest Sharpe ratio weight solutions. [19]

The calculation of the Single Index Model is based on the return, beta of each stock.

$$E(R_p) = \sum_{i=1}^7 w_i \alpha_i + \sum_{i=1}^7 w_i \beta_i E(R_M) \quad (1)$$

$$\sigma_p^2 = (\sum_{i=1}^7 w_i \beta_i)^2 \sigma_M^2 + \sum_{i=1}^7 w_i^2 \sigma_{ei}^2 \quad (2)$$

$$Sharpe = \frac{E(R_p)}{\sigma_p} \quad (3)$$

From equation (1), (2), (3), $E(R_p)$ is the expected return of the portfolio, and w_i refers to the weight of the stock i . α_i and β_i respectively refers to the alpha and beta from the index model. σ_p^2 is the variance of the portfolio, while σ_M^2 refers to the variance of the market and σ_{ei}^2 means the variance or risk level of the error term. *Sharpe* refers to the sharpe ratio of the portfolio.

Then this paper considers the above constraints that has been proposed before to establish different portfolio in the following part.

4.1 Constraint 1

Table 2 illustrated the optimal portfolio weighted allocation under constraint 1, when $\sum |W_i| \leq 2$, the optimal portfolio weighted allocation. The results have suggested to allocate -50% of weight on SPX, nearly 23% on MSFT, 0.73% on ORCL, 7.98% on JPM, around 33% on BRK/A, 49% on PG and 36% on JNJ. This optimal portfolio offered a weighted return of 10.90%, with a standard deviation of 12.54%.

Table 2. Optimal Portfolio Solutions of Constraint 1 under Index Model

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
W's	-50.00%	23.06%	0.73%	7.98%	33.02%	49.57%	35.64%
Alpha's	0.00%	6.40%	0.53%	2.27%	4.96%	6.79%	4.69%
Beta's	100.00%	106.69%	108.75%	138.09%	53.67%	39.25%	52.76%
Expected Market Return	6.16%	6.16%	6.16%	6.16%	6.16%	6.16%	6.16%
Expected Market SD	15.10%	15.10%	15.10%	15.10%	15.10%	15.10%	15.10%
SD of residuals	0.00%	20.97%	24.18%	21.65%	14.18%	13.68%	13.16%
Expected Portfolio Return				10.90%			
Expected Portfolio SD				12.54%			
Sharpe Ratio(annualized)				0.8692			

4.2 Constraint 2

Table 3 illustrated the optimal portfolio weighted allocation under the constraint $|W_i| \leq 1$. The results have suggested to allocate -61.29% of weight on SPX, nearly 25% on MSFT, 1.79% on ORCL, 10.05% on JPM, around 35% on BRK/A, 51% on PG and 38% on JNJ. This optimal portfolio offered a weighted return of 11.31%, with a standard deviation of 12.99%. Comparing to the weighted return from constraint 1, the weighted return from constraint 2 is larger while sacrificing some standard deviation.

Table 3. Optimal Portfolio Solutions of Constraint 2 under Index Model

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
W's	-61.29%	25.19%	1.79%	10.05%	34.97%	51.53%	37.76%
Alpha's	0.00%	6.40%	0.53%	2.27%	4.96%	6.79%	4.69%
Beta's	100.00%	106.69%	108.75%	138.09%	53.67%	39.25%	52.76%
Expected Market Return	6.16%	6.16%	6.16%	6.16%	6.16%	6.16%	6.16%
Expected Market SD	15.10%	15.10%	15.10%	15.10%	15.10%	15.10%	15.10%
SD of residuals	0.00%	20.97%	24.18%	21.65%	14.18%	13.68%	13.16%
Expected Portfolio Return	11.31%						
Expected Portfolio SD	12.99%						
Sharpe Ratio(annualized)	0.8708						

4.3 Constraint 3

Table 4 depicted the optimal portfolio weighted allocation free of constraint. The results have suggested to allocate -61.29% of weight on SPX, nearly 25% on MSFT, 1.79% on ORCL, 10.05% on JPM, around 35% on BRK/A, 51% on PG and 38% on JNJ, which was the exact same portfolio allocation as the portfolio allocation under constraint 2. This optimal portfolio offered a weighted return of 11.31%, with a standard deviation of 12.99%. Comparing to the weighted return from constraint 1, the weighted return from constraint 2 is larger while sacrificing some standard deviation.

Table 4. Optimal Portfolio Solutions of Constraint 3 under Index Model

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
W's	-61.29%	25.19%	1.79%	10.05%	34.97%	51.53%	37.76%
Alpha's	0.00%	6.40%	0.53%	2.27%	4.96%	6.79%	4.69%
Beta's	100.00%	106.69%	108.75%	138.09%	53.67%	39.25%	52.76%
Expected Market Return	6.16%	6.16%	6.16%	6.16%	6.16%	6.16%	6.16%
Expected Market SD	15.10%	15.10%	15.10%	15.10%	15.10%	15.10%	15.10%
SD of residuals	0.00%	20.97%	24.18%	21.65%	14.18%	13.68%	13.16%
Expected Portfolio Return	11.31%						
Expected Portfolio SD	12.99%						
Sharpe Ratio(annualized)	0.8708						

4.4 Constraint 4

Table 5 depicted the optimal portfolio weighted allocation under constraint that $w_i \geq 0$. The results have suggested to allocate 0 weight to SPX, 10% on MSFT, 0% on ORCL, 0% on JPM, around 20% on BRK/A, 40% on PG and 20% on JNJ. This optimal portfolio offered a weighted return of 9.11%, with a standard deviation of 11.30%. Comparing to the weighted return from constraint 1,2 and 3, the weighted return from constraint 4 is much smaller while its standard deviation is smaller as well.

Table 5. Optimal Portfolio Solutions of Constraint 4 under Index Model

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
W's	0.00%	10.00%	0.00%	0.00%	20.00%	40.00%	20.00%
Alpha's	0.00%	6.40%	0.53%	2.27%	4.96%	6.79%	4.69%
Beta's	100.00%	106.69%	108.75%	138.09%	53.67%	39.25%	52.76%
Expected Market Return	6.16%	6.16%	6.16%	6.16%	6.16%	6.16%	6.16%
Expected Market SD	15.10%	15.10%	15.10%	15.10%	15.10%	15.10%	15.10%
SD of residuals	0.00%	20.97%	24.18%	21.65%	14.18%	13.68%	13.16%
Expected Portfolio Return	9.11%						
Expected Portfolio SD	11.30%						
Sharpe Ratio(annualized)	0.8066						

4.5 Constraint 5

Table 6 depicted the optimal portfolio weighted allocation by deliberately allocating zero weight to SPX to see its effect on the portfolio return. The optimal results have suggested to allocate 0 weight to SPX, 20% on MSFT, 0% on ORCL, 0% on JPM, around 20% on BRK/A, 40% on PG and 30% on JNJ. This optimal portfolio offered a weighted return of 9.20%, with a standard deviation of 11.34%. Comparing to the weighted return from constraint 1,2 and 3, the weighted return from constraint 4 is much smaller while its standard deviation is smaller as well. The maximum weight is held by PG, which accounts for 40%. Thus, compared with the no constraint portfolio, SPX500 still has remarkable influence in portfolio construction.

Table 6. Optimal Portfolio Solutions of Constraint 5 under Index Model

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
W's	0.00%	20.00%	0.00%	00.00%	20.00%	40.00%	30.00%
Alpha's	0.00%	6.40%	0.53%	2.27%	4.96%	6.79%	4.69%
Beta's	100.00%	106.69%	108.75%	138.09%	53.67%	39.25%	52.76%
Expected Market Return	6.16%	6.16%	6.16%	6.16%	6.16%	6.16%	6.16%
Expected Market SD	15.10%	15.10%	15.10%	15.10%	15.10%	15.10%	15.10%
SD of residuals	0.00%	20.97%	24.18%	21.65%	14.18%	13.68%	13.16%
Expected Portfolio Return	9.20%						
Expected Portfolio SD	11.34%						
Sharpe Ratio(annualized)	0.8116						

4.6 Minimum Risk Portfolio and the compare of five constraints

Table 7 depicted the minimal risk portfolio weighted allocation. The results have suggested to allocate 50% weight to SPX, -10% on MSFT, 0% on ORCL, -20% on JPM, around 20% on BRK/A, 30% on PG and 30% on JNJ. With this weights allocated, the minimal risk of 10.51% was achieved.

Table 7. Minimum Risk Portfolio

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
W's	50.00%	-10.00%	0.00%	-20.00%	20.00%	30.00%	30.00%
Alpha's	0.00%	6.40%	0.53%	2.27%	4.96%	6.79%	4.69%
Beta's	100.00%	106.69%	108.75%	138.09%	53.67%	39.25%	52.76%
Expected Market Return	6.16%	6.16%	6.16%	6.16%	6.16%	6.16%	6.16%
Expected Market SD	15.10%	15.10%	15.10%	15.10%	15.10%	15.10%	15.10%
SD of residuals	0.00%	20.97%	24.18%	21.65%	14.18%	13.68%	13.16%
Expected Portfolio Return	7.41%						
Expected Portfolio SD	10.51%						
Sharpe Ratio(annualized)	0.7045						

Figure 3 has summarized all the MRF and EF for each constraint together in one graph. Through comparison, under the same standard deviation level, the portfolio under no constraint offers the highest expected return. The possible reason behind may be that under no constraint, much more aggressive short position is available which may bring greater return.

After the risk level reaches around 20%, both constraint 1 and constraint 4 fail to come up with solutions. This may be due to both constraint 1 and constraint 4 offers stricter weight allocation requirement or strict short position, and portfolios with risk higher than 20% may not be constructed under the two constraints.

The capital allocation line converges with the efficient frontier with no constraint on the point where the standard deviation was around 12.99% with the return around 11.31%. The Efficient Frontier and the Minimal Return Frontier intersects on the minimal risk point which gives a standard deviation of around 10.51% with an expected return of 7.41%.

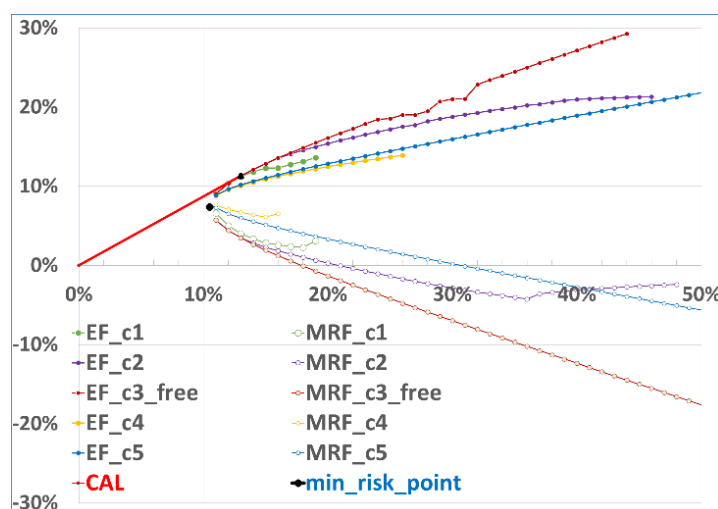


Fig. 3 Summary of EF, MRF, CAL, minimal risk point and maximum Sharpe ratio point under five constraints, Index Model

5. Markowitz Model

To build a portfolio of the six companies and SPX500, this section introduces the Markowitz model, which also includes five constraints to compare the difference of each building portfolio. Constraint three is the no constraint portfolio, so this is a benchmark solution. This section still take the Minimal Risk Portfolio, Maximal Sharpe Ratio, and efficient frontier into consideration.

5.1 Markowitz model

The Markowitz model is a portfolio optimization model to determine the risk-return opportunities available to the investors. These are summarized by the minimum-variance frontier of the risky assets. This frontier can be described as a graph of the lowest possible variance that can be attained for a given portfolio. This project used the data of the monthly excess return, variances, and covariances to calculate the Minimal Risk Portfolio, Maximal Sharpe Ratio, Efficient Frontier from constraint 1 to constraint 5. Constraint 3 is also used as a benchmark to print graph 1 which consists of two points just mentioned and the Capital Allocation Line, Efficient Frontier, and Minimum Variance Frontier.

The calculation of the Markowitz model is based on the return, standard deviation of each stock, and the correlation matrix.

$$R = \sum_{i=1}^7 w_i \times r_i \quad (4)$$

$$Stdev = ((S_1, S_2 \dots S_i) * (w_1, w_2 \dots w_i) \begin{pmatrix} r_{11} & \dots & r_{1i} \\ \vdots & \ddots & \vdots \\ r_{i1} & \dots & r_{ii} \end{pmatrix} (S_1, S_2 \dots S_i)^T * (w_1, w_2 \dots w_i)^T)^{\frac{1}{2}} \quad (5)$$

$$Sharpe = \frac{R}{Stdev} \quad (6)$$

In those three equations, R is the return of the portfolio, w_i is the weight of stock i , r_i is the yearly average return of stock i . $Stdev$ is the standard deviation of the portfolio. S_1 is the yearly standard deviation of the stock i , r_{ij} is the correlation between stock i and stock j . And using return divide by standard deviation as the Sharpe ratio equation. When calculating the Efficient Frontier, the Solver table is used to calculate them. This study set the minimum value to 0, set the maximum value to 0.7, and set the increment to 0.01.

In the following, it is going to use these formulas to further analyze the portfolios which are based on the Markowitz model on each constraint.

5.2 Constraint 1

Table 8 shows the portfolio base on constraint 1, which is: $\sum_{i=1}^7 |w_i| \leq 2$. In this constraint, the building of these portfolios is limited by Regulation T. From this table, the return of Minimal Risk Portfolio is 7.79% and the standard deviation is 11.49%. Based on this allocation, PG has the maximum weight, which is about 34%, and JPM has the only negative weight, which is about -9%. The Maximal Sharpe Ratio is 0.8633, and this portfolio has 11.74% as its return and 13.60% as its standard deviation. PG

still has the maximum weight, which is 55%, the minimum weight belongs to SPX, which has -50% as its weight.

Compared with Table 10, it has the same Minimal Risk Portfolio, whereas the Maximal Sharpe Ratio under this constraint is slightly lower than the no constraint portfolio. Therefore, limitation on the total absolute value will cause this performance.

Table 8. Optimal Portfolio Solutions of Constraint 1 under Markowitz Model

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
Minimal Risk Portfolio	0.28	0.07	0.02	-0.09	0.25	0.34	0.14
Maximal Sharpe Ratio	-0.5	0.35	0.04	0.09	0.3	0.55	0.18
Return (MRP)	7.79%						
StDev (MRP)	11.49%						
Sharpe (MRP)	0.6779						
Return (MSR)	11.74%						
StDev (MSR)	13.60%						
Sharpe (MSR)	0.8633						

5.3 Constraint 2

Table 9 shows the portfolio base on constraint 2, $|w_i| \leq 1$ for all i . This constraint is based on the tendency that investors will not make too many transactions in one single stock. The Minimal Risk Portfolio has a return of 7.79% and a standard deviation of 11.49%. Based on this allocation, PG has the maximum weight, which is about 34%, and JPM has the only negative weight, which is about -9%.

The Maximal Sharpe Ratio is 0.870 when the return is 12.85% and the standard deviation is 14.67%. PG still has the maximum weight, which is 61%, the minimum weight belongs to SPX, which has -79% as its weight.

Both Minimal Risk Portfolio and Maximal Sharpe Ratio are the same as the portfolios with no constraint. So, constraint 2 has a little limitation when builds portfolios.

Table 9. Optimal Portfolio Solutions of Constraint 2 under Markowitz Model

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
Minimal Risk Portfolio	0.28	0.07	0.02	-0.09	0.25	0.34	0.14
Maximal Sharpe Ratio	-0.79	0.41	0.07	0.14	0.33	0.61	0.23
Return (MRP)	7.79%						
StDev (MRP)	11.49%						
Sharpe (MRP)	0.6779						
Return (MSR)	12.85%						
StDev (MSR)	14.76%						
Sharpe (MSR)	0.8709						

5.4 Constraint 3

Table 10 shows the portfolio based on constraint 3 which means no constraint, and this is the basic form of the Markowitz model. The Minimal Risk Portfolio has a return of 7.79% and a standard deviation of 11.49%. Based on this allocation, PG has the maximum weight, which is about 34%, and JPM has the only negative weight, which is about -9%. From table 1, PG has the lowest yearly standard deviation compared with the other six stocks, which is 14.90%, while JPM has the highest yearly standard deviation compared with the other six stocks, which is 30.06%. So based on the no constraint, the maximum and minimum standard deviation result in maximum and minimum weight in allocation.

Table 10. Optimal Portfolio Solutions of Constraint 3 under Markowitz Model

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
Minimal Risk Portfolio	0.28	0.07	0.02	-0.09	0.25	0.34	0.14
Maximal Sharpe Ratio	-0.79	0.41	0.07	0.14	0.33	0.61	0.23
Return (MRP)	7.79%						
StDev (MRP)	11.49%						
Sharpe (MRP)	0.6779						
Return (MSR)	12.85%						
StDev (MSR)	14.76%						
Sharpe (MSR)	0.8709						

The Maximal Sharpe Ratio is 0.870 when the return is 12.85% and the standard deviation is 14.67%. PG has the maximum weight, which is 61%, the minimum weight belongs to SPX, which has -79% as its weight. From table 1, SPX has the lowest yearly return compared with the other six stocks, which is 6.16%, and PG has the lowest yearly standard deviation compared with the other six stocks. The characteristic of these two stocks has a great influence on allocation based on no constraint condition.

5.5 Constraint 4

Table 11 shows the portfolio base on constraint 4, which is $w_i \geq 0$ for all i . This constraint is based on the security of the short sale, and this constraint does not allow to take a short position. Because of this constraint, the Minimal Risk Portfolio has a bigger return and standard deviation than the no constraint situation, the return is 8.38% and the standard deviation is 11.66%. Based on this allocation, PG has the maximum weight, which is about 36%, and JPM has the lowest weight, which is about 0%.

The Maximal Sharpe Ratio is 0.7991 and becomes smaller than no constraint situation, the return is 9.88% and the standard deviation is 12.36%. There are three stocks in the same 0% weights, which are SPX, ORCL, and JPM. PG has the maximum weight, which is 46%.

So, compared with the no constraint portfolio, $w_i \geq 0$ has a great influence on both the minimal risk portfolio and maximal Sharpe ratio allocations.

Table 11. Optimal Portfolio Solutions of Constraint 4 under Markowitz Model

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
Minimal Risk Portfolio	0.14	0.07	0.02	0.00	0.23	0.36	0.17
Maximal Sharpe Ratio	0.00	0.25	0.00	0.00	0.22	0.46	0.07
Return (MRP)	8.38%						
StDev (MRP)	11.66%						
Sharpe (MRP)	0.7186						
Return (MSR)	9.88%						
StDev (MSR)	12.36%						
Sharpe (MSR)	0.7991						

5.6 Constraint 5

Table 12 shows the portfolio base on constraint 5, which is $w_1 = 0$. This examines the impact of the index and the change of return and standard deviation. Without the influence of the SPX500, Maximal Sharpe Ratio decreases to 0.8003, its return is 9.95% and its standard deviation is 12.44%. Besides the influence of the SPX, ORCL has the minimum weight of -2%, which is the different stock with no constraint portfolio, because ORCL has the lowest yearly standard deviation in stocks except SPX. The maximum weight is still held by PG, which is about 46%.

The Minimal Risk Portfolio has a return of 8.77% and a standard deviation of 11.68%. Compared with the no constraint allocation, it has a higher standard deviation and lower return. Besides the influence of the SPX, PG still has the maximum weight, which is about 39%, and JPM still has the minimum weight, which is about -4%. Compared with the no constraint portfolio, the weights of these two stocks are higher.

So, compared with the no constraint portfolio, SPX500 still has remarkable influence during the portfolio construction.

Table 12. Optimal Portfolio Solutions of Constraint 5 under Markowitz Model

	SPX	MSFT	ORCL	JPM	BRK/A	PG	JNJ
Minimal Risk Portfolio	0.00	0.11	0.05	-0.04	0.29	0.39	0.19
Maximal Sharpe Ratio	0.00	0.26	-0.02	0.00	0.23	0.46	0.08
Return (MRP)	8.77%						
StDev (MRP)	11.68%						
Sharpe (MRP)	0.7513						
Return	9.95%						

(MSR)	
StDev	12.44%
(MSR)	
Sharpe	0.8003
(MSR)	

5.7 Compare of five constraints

The Minimal Risk Portfolio point, Maximal Sharpe Ratio portfolio, and Capital Allocation Line are put in the figure, all of them are based on constraint 3. Additionally, this study put Efficient Frontier and Minimum Return Frontier which are based on each constraint in Figure 4.

In this figure, between different constraints, there are remarkable differences. These five constraints can be separated into three groups, constraint 1&2, constraint 3, and constraint 4&5. Constraint3 has the best performance, it has the highest expected return based on each given standard deviation. Constraint 1&2 has lower return, and the distances between constraint 3 have increased through the increase of the standard deviation. Constraint 4&5 has a similar changing trend as constraint 1&2. however, Constraint 4&5 has a stronger limitation on allocation. It can be easily identified that after when the standard deviation is higher than about 25%, there will be no allocation to meet the conditions on constraints 4&5.

Table13. Portfolio with different constraints under Markowitz model and Index model

		SPX	MSFT	ORCL	JPM	BRK/A	P&G	JNJ	EPR	St.D	Sharpe ratio
Index model	Constraint1	-50.00%	23.06%	0.73%	7.98%	33.02%	49.57%	35.64%	10.90%	12.54%	0.8692
	Constraint2	-61.29%	25.19%	1.79%	10.05%	34.97%	51.53%	37.76%	11.31%	12.99%	0.8708
	Constraint3	-60.00%	30.00%	0.00%	10.00%	30.00%	50.00%	40.00%	11.31%	12.99%	0.8708
	Constraint4	0.00%	10.00%	0.00%	0.00%	20.00%	40.00%	20.00%	9.11%	11.30%	0.8066
	Constraint5	0.00%	20.00%	0.00%	0.00%	20.00%	40.00%	30.00%	9.20%	11.34%	0.8116
Markowitz model	Constraint1	-50.00%	40.00%	0.00%	10.00%	30.00%	50.00%	20.00%	11.74%	13.60%	0.8633
	Constraint2	-80.00%	40.00%	10.00%	10.00%	30.00%	60.00%	20.00%	12.85%	14.76%	0.8709
	Constraint3	-80.00%	40.00%	10.00%	10.00%	30.00%	60.00%	20.00%	12.85%	14.76%	0.8709
	Constraint4	0.00%	30.00%	0.00%	0.00%	20.00%	50.00%	10.00%	9.88%	12.36%	0.799
	Constraint5	0.00%	30.00%	0.00%	0.00%	20.00%	50.00%	10.00%	9.95%	12.44%	0.8003

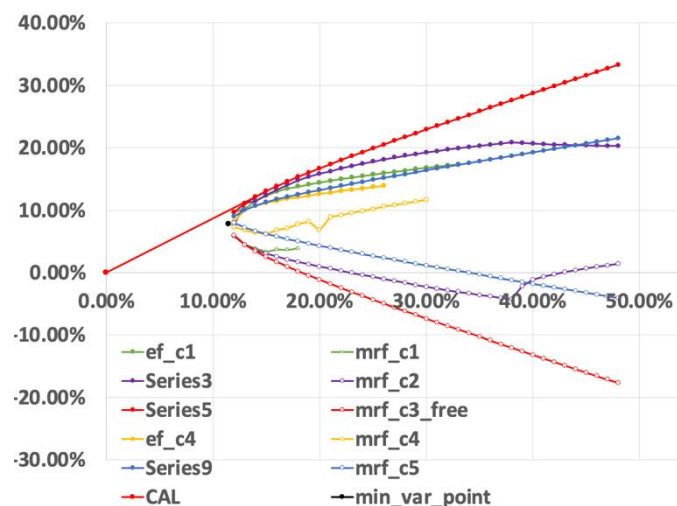


Fig. 4 EF, MRF, CAL, Minimal Risk, Maximum Sharpe Ratio under five constraints, Markowitz model

6. Discussion

This part compares the optimal portfolio under the two models with different constraint conditions and also contrast some parameters of different models under the same constraints. Based on these, the significance of management has been obtained.

Based on the above analysis, this paper calculates the weight of the investment portfolios constructed by two different models under 5 constraints and the corresponding investment information of the portfolios, and plot the results as Table 13.

In Table 13, by comparing different models and weights, it has been found that there is still necessity to short the S&P500 in order to have more capital to buy other stocks, since predicting it will go down in the future. And as for Microsoft, Berkshire Hathaway, Jonson & Jonson and P&G, higher weights in every portfolio than other two companies have been given.

For returns and standard deviations, by comparison, it is found that higher returns are often accompanied by higher standard deviations, which means higher risk. Just looking at the results, the higher return portfolio will also have a relatively high Sharp ratio, but it's not the necessary case in reality. The portfolio under constraints 2 and 3 has a higher expected return and Sharpe ratio. Therefore, constraints 2 and 3 are suitable portfolios for risk-neutral investors, and constraints 4 and 5 are more suitable for risk-averse investors.

This project found both models are extreme in terms of the number of parameters and based on certain assumptions. Therefore, both of them have lots of defects. Only looking at Table 13, they have great differences for the optimal portfolios under different constraints. But they generally have a high degree of consistency, so they have certain reference significance for reality.

According to the above analysis, a series of managerial suggestions for investors has been proposed.

(1) A variety of different asset portfolios can help us effectively reduce non-systemic risks, and both Markowitz model and Index model help investors find the optimal investment ratio. Whatever for individual investors and institutional investors, it's no doubt that those models can effectively help investors make better investments.

(2) Markowitz model is the basis of the Index model, which can be used more accurately Risk management for a small number of asset classes. And the index model is an empirical improvement of the Markowitz model, which greatly simplifies the calculation steps and requires only a few parameters to obtain the best asset portfolio, so it has incomparable advantages over the Markowitz model in the analysis of a large number of securities portfolios.

(3) The establishment of Markowitz's portfolio theory has brought rapid development to the financial field, and on this basis a variety of portfolio models have been produced including Index model. The establishment of these theories provides a theoretical basis for investors.

7. Conclusion

This paper tries to use Markowitz Model (MM) and Index Model (IM) to analyze the behavior of optimal portfolio through the stock data in the past ten years. This project starts from stock data of six companies, including MSFT, ORCL, JPM, BRK/A, PG and JNJ. They are from three different industries: technology, financial services and health care.

The Markowitz model (MM model) is a portfolio optimization model. By analyzing possible portfolios of given securities, the MM model could be used to select the most efficient portfolio. Specifically, by choosing securities with different moving directions, the MM model shows investors how to reduce their risk. The single-index model is a simple asset pricing model to measure both the risk and the return of a stock. The factor can be represented by the rate of return on a market index, such as the S&P 500. By using the single-index model, fewer necessary inputs is need to derive the Optimal Portfolio for Asset Allocation, comparing with the Markowitz Portfolio Selection Procedure. It has been found that minimum risk portfolio, maximum return and maximum Sharpe ratio under different scenarios. Five constraints have been added, and these constraints are regulation T, arbitrary "box" constraints, free, open-ended mutual fund, and test if including an index will have an impact.

By comparing two models, this study found both models provide the best portfolio solution with the highest Sharpe ratio in constraint 2, 3. The possible rationale behind may be that both these constraints allow taking relatively radical short position to achieve a higher return. This project found

relatively similar results from both of the two models. By comparing the calculation procedure of the two models, the single index model was considered simpler in calculation.

After viewing the whole process of the research, it has been found that there are some limitations in this research. For both MM and IM models, they have some assumptions. They assume the investors would have all data available and be rational. But in real situations, it is not as ideal. The stock price would be affected by other factors not mentioned in the model. So, the result may be slightly deviate from the ideal models. For further research, other financial models will be considered, such as momentum model, and compare with MM and IM. In the future, it will be possible to find out the similarities and differences between these models and view detailed behaviors of them under different constraints.

References

- [1] Bonfim, D. (2009). Credit risk drivers: Evaluating the contribution of firm level information and of macroeconomic dynamics. *Journal of Banking & Finance*, 33(2), 281-299.
- [2] Ivanova, M., & Dospatliev, L. (2017). Application of Markowitz portfolio optimization on Bulgarian stock market from 2013 to 2016. *International Journal of Pure and Applied Mathematics*, 117(2), 291-307.
- [3] Huang, X., & Qiao, L. (2012). A risk index model for multi-period uncertain portfolio selection. *Information Sciences*, 217, 108-116.
- [4] Sarwar, S., Shahbaz, M., Anwar, A., & Tiwari, A. K. (2019). The importance of oil assets for portfolio optimization: The analysis of firm level stocks. *Energy Economics*, 78, 217-
- [5] Wood, David. (2002). Portfolio Optimization Benefits from Integrating Analysis of Risk, Strategy, and Valuation. *Oil and Gas Journal*. 100. 26-33.
- [6] Brinson, G. P., Hood, L. R., & Beebower, G. L. (1986). Determinants of portfolio performance. *Financial Analysts Journal*, 42(4), 39-44.
- [7] Fama, E. F., & French, K. R. (1993). Common risk factors in the returns on stocks and bonds. *Journal of financial economics*, 33(1), 3-56.
- [8] Asness, C. S., Frazzini, A., & Pedersen, L. H. (2019). Quality minus junk. *Review of Accounting Studies*, 24(1), 34-112.
- [9] Lev, B., & Srivastava, A. (2019). Explaining the recent failure of value investing. NYU Stern School of Business.
- [10] Markowitz, H. M. (1991). Foundations of portfolio theory. *The journal of finance*, 46(2), 469-477.
- [11] Pogue, G. A. (1970). An extension of the Markowitz portfolio selection model to include variable transactions' costs, short sales, leverage policies and taxes. *The Journal of Finance*, 25(5), 1005-1027.
- [12] Soleimani, H., Golmakani, H. R., & Salimi, M. H. (2009). Markowitz-based portfolio selection with minimum transaction lots, cardinality constraints and regarding sector capitalization using genetic algorithm. *Expert Systems with Applications*, 36(3), 5058-5063.
- [13] Sharpe, W. F. (1963). A simplified model for portfolio analysis. *Management science*, 9(2), 277-293.
- [14] Hristache, M., Juditsky, A., & Spokoiny, V. (2001). Direct estimation of the index coefficient in a single-index model. *Annals of Statistics*, 595-623.
- [15] Bilbao, A., Arenas, M., Jiménez, M., Perez Gladish, B., & Rodríguez, M. V. (2006). An extension of Sharpe's single-index model: portfolio selection with expert betas. *Journal of the Operational Research Society*, 57(12), 1442-1451.
- [16] Margin account requirements. (n.d.). Retrieved from <https://www.finra.org/rules-guidance/key-topics/margin-accounts>
- [17] 15 U.S. Code § 80a-12 - functions and activities of investment companies. (n.d.). Retrieved from <https://www.law.cornell.edu/uscode/text/15/80a-12>
- [18] Kenton, W. Beta. Retrieved from <https://www.investopedia.com/terms/b/beta.asp>
- [19] Mandal, N. (2013). Sharpe's single index model and its application to construct optimal portfolio: an empirical study. *Great Lake Herald*, 7(1), 1-19.