Options Pricing Comparison between the Black-Scholes Model and the Binomial Tree Model: A Case Study of American Equity Option and European-style Index Option

Yawei Wu*

[1]Northview High School, Georgia, United States

*Corresponding author: bewu@hmc.edu

Abstract. In recent years, quantitative researchers used a wide range of models to price options, from the Black-Scholes model to more complex models such as the Heston model. This paper aims to analyze the effectiveness of the Black-Scholes model and the Binomial Tree model by using them to price Berkshire Hathaway's equity options and European-style S&P 100 index options. The method used in this paper is gathering the market data of the options first. Second, using the data gathered to price the options by applying the Black-Scholes and Binomial Tree models. Third, comparing the derived theoretical price with the market price by getting the Sum of Square Errors. Lastly, determining the best model for each type of option. Through this research, the author found that comparing the two models, the Binomial Tree model derives a smaller Sum of Square Errors when pricing European-style index options, and the Black-Scholes model derives a smaller Sum of Square Errors when pricing American equity options. Thus, the Binomial Tree model is a better model to price European-style index options, and the Black-Scholes model is more effective when pricing American equity options.

Keywords: Option pricing; Equity Option; Index Option; Black-Scholes model; Binomial Tree model.

1. Introduction

Options are financial derivatives that grant buyers the right to exercise the contract at a promised price within a predetermined period. Usually, option contracts trade underlying securities such as stocks or commodities. Options are categorized into two types: vanilla and non-vanilla. Vanilla options refer to call and put options that do not have unusual features, while non-vanilla options are options that have various features. Some common vanilla options are European and American options. Look-back, barrier, and chooser options are examples of non-vanilla options. This paper will focus on pricing European and American options. Options that can only be exercised on the expiration date are European options, while options that can be exercised before or on the expiration day are American options.

The purpose of pricing options is to derive the theoretical price of the options by using different variables from the market. Two pioneers who developed a pricing model for options were Black and Scholes. Together, they introduced the widely-used Black-Scholes model (B-S model) in 1973 [1]. To get the theoretical price, the B-S model requires five inputs: option strike price, underlying asset's current price, time to maturity, interest rate, and volatility. Since then, more complicated models were developed to price various styles of options. In 1979, the Binomial Tree model (BT model), also known as the Cox-Ross-Rubinstein model, was a simplified approach to price options presented by Cox, Ross, and Rubinstein [2]. This model has similar assumptions as the B-S model, but it approximates the price in discrete time, which differs from Black-Scholes' continuous approach. As the number of binomial steps increases, the binomial distribution would approach the Black-Scholes' log-normal distribution [3, 4]. So, the B-S model is derived when the number of binomial steps approaches infinity.

Despite its popularity, the B-S model has some dubious assumptions. The model assumed that the return of the underlying is normally distributed with known mean and variance, and the implied volatilities for all strike prices are constant [5]. In real life, the implied volatilities are not constant, and the variance of the return is randomly distributed. To solve these issues, researchers introduced new mathematical models, leading to more accurate option pricing. In 1976, Merton developed a jump-diffusion model for price options [6]. A stochastic, meaning random, process that has discrete movements is called a jump process. In 1977, Boyle used the Monte Carlo simulation, which allows random variance, to price options [7]. In 1993, Steven Heston revealed the Heston model [8], which assumes the underlying asset has stochastic volatility.

Because there are various pricing models investors can use, many researchers wrote papers to compare these option pricing models. This paper will compare the effectiveness of the B-S model and the BT model by using them to price Berkshire Hathaway's equity options and European-style S&P 100 index options. So, a case study was conducted that includes the following steps: gathering option data from Yahoo! Finance, calculating theoretical prices using different models, then using the Excel spreadsheet's Add-in Solver function to calibrate the optimal volatility by minimizing the Sum of Square Errors (SSE), also known as the Sum of Square Residuals (SSR) [9].

The result shows that the BT model derives a smaller SSE than the B-S model for the European-style S&P 100 index options. As for Berkshire Hathaway's equity options, the B-S model results in a smaller SSE than the BT model. Although both models are easy-to-use, it is suggested to use the BT model when pricing European-style index options and the B-S model when pricing American equity options with no dividends.

The rest of this paper is divided into four sections. Demonstration of gathered data and introduction to the pricing models are in Section 2. Section 3 exhibits the results of the research. Section 4 further analyzes the result. Section 5 summarizes this research.

2. Data and Methodology

2.1 Data

In this paper, the European-style S&P 100 index options (Symbol: ^XEO) will be used to represent European-style index options. Although the option is based on the U.S. Standard & Poor 100 index, it is traded in European style. So, it is treated as a European index option in U.S. currency. Berkshire Hathaway's options (Symbol: BRK-B) will be used as a representative of non-dividend-paying American equity options. The European-style S&P 100 index options were chosen because most indexes use European options, including the S&P 100 index, one of the most popular indexes on U.S. exchanges. And Berkshire Hathaway's equity options were chosen because Berkshire Hathaway Inc. is one of the most popular value stocks. Warren Buffet is the current owner of this conglomerate holding company, which holds more than 10 companies from different industries. Both options could represent the current U.S. market.

All calculations in this paper are based on data gathered from Yahoo! Finance. To price options using the B-S model, variables including the options' strike prices, time to maturity, underlying asset's price, interest rate, and volatility are needed. The strike price, K, is the amount of money the buyer gets or the seller pays when the option exercises. Time to maturity, T, is the time remaining until the maturity date. S is the current price of the underlying asset. r is the risk-free interest rate. σ is volatility, the rate at which the underlying asset experiences price changes. Higher volatility often indicates higher risk. To price using the BT model, the same five variables required in the B-S model are needed as well. To perform calibration, the market prices of the options are required.

On June 31, 2022, the European-style S&P 100 index (^XEO) had an S of 1725.61. The maturity time was selected as December 16, 2022, meaning there are 118 market days until maturity. In this paper, the interest rate is 1.5 percent, the U.S. interest rate as of June 16, 2022. This paper uses five call and five put options with strike prices 1700, 1720, 1740, 1760, and 1780. The strike prices are selected because they are near the S. The European-style S&P 100 index options have market prices shown below.

Table 1. Market prices of ^XEO options corresponding to their strike prices

Calls	
Strike prices	Market prices
1700	108.5
1720	13.2
1740	402
1760	106.5
1780	132
Puts	
Strike prices	Market prices
1700	148.11
1720	152.61
1740	675.5
1760	702
1780	694

On June 21, 2022, Berkshire Hathaway Inc.'s stock(Symbol: BRK-B) had an *S* of 273.85. The maturity time was selected as July 15, 2022, meaning there are 18 market days until maturity. The interest rate is identical to the interest rate used in the European-style S&P 100 index options, which is 1.5 percent. This paper uses five call and five put options with strike prices 260, 265, 270, 275, and 280. The strike prices are selected because they are near the *S*. The market prices of the BRK-B options are as follows.

Table 2. Market prices of BRK-B options corresponding to their strike prices

Calls	
Strike prices	Market prices
260	15.15
265	11.9
270	8.6
275	5.8
280	3.45
Puts	
Strike prices	Market prices
260	3.1
265	4.24
270	6.2
275	8.59
280	11.05

2.2 Methodology

2.2.1 Black-Scholes Model

A path-breaking mathematical method that was popularly used to price options was the B-S model. As mentioned in Section 1, the model requires five inputs: the underlying asset's current price, strike price, time to expire, interest rate, and volatility. Black and Scholes indicated that they made several major assumptions when developing the model [1]:

The option can only be exercised on the expiration date, which means it's European-style.

When trading the options, there are no transaction fees.

The underlying asset does not pay dividends or any other distributions.

Throughout the life of the options, the underlying's risk-free rate and volatility are known and constant.

Markets are efficient and random, meaning market movements are unpredictable.

The returns of the underlying are log-normally distributed.

The B-S formula is a solution of the B-S equation, a partial differential equation (PDE). The B-S PDE came from the stochastic differential equation (SDE). From the assumptions, it is known that the price of the underlying asset follows geometric Brownian motion, which is

$$dS = \mu S dt + \sigma S dW \tag{1}$$

$$\frac{dS}{S} = \mu dt + \sigma dW \tag{2}$$

where *W* is a Brownian motion, σ is the volatility, *S* is the underlying asset, and the rate of expected return is μ. Then, from Ito's Lemma [10], the B-S PDE is

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial V}{\partial S} - rC = 0 \tag{3}$$

where the option price is *C*, expressed as a function of *S*, the stock price, at time *t*, and the risk-free interest rate is *r*. Then, the Feynman-Kac formula [11] can be used to derive a solution to Equation (2), the Black Scholes equation. The method used in the formula is also known as the martingale risk-neutral pricing. From the Feynman-Kac formula, the B-S formula is derived, which is

$$C(t, S(t)) = S(t)N(d1) - Ke^{-r(T-t)}N(d2) \tag{4}$$

$$P(t, S(t)) = Ke^{-r(T-t)}N(-d2) - S(t)N(-d1) \tag{5}$$

where

$$N(x) := P[Z \le x] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\left(-\frac{y^2}{2}\right)} dy \tag{6}$$

$$d1 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \tag{7}$$

$$d2 = \frac{1}{\sigma\sqrt{T-t}}\left[\log\left(\frac{S(t)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right] = d1 - \sigma\sqrt{T-t} \tag{8}$$

There are more detailed processes. Hull demonstrated the whole derivation process in his book "Options, Futures, and Other Derivatives" [5]. Pricing the American equity options does not involve modifying the B-S formula because Merton showed that when rational investors found out option value is the greatest if exercised on the maturity date, American-style options would value the same as European-style options [12].

2.2.2 Binomial Tree model

The Cox-Ross-Rubinstein model, or the BT model, is a simple discrete-time numerical method to value options that optimizes premature exercises, such as the American options. Sharpe was the first to suggest discrete-time option pricing [13]. Later, Cox, Ross, and Rubinstein formalized the model in 1979. The formulas below serve as the foundation of this case study.

Calculating the binomial option price involves generating the stock prices, getting option prices at each final node, and working backward to find the option price at the first node. The BT model assumed in every step of the tree, the underlying asset will either move up or down by a factor, denoted as *u* for up and *d* for down. *u* and *d* must satisfy $u \ge 1$ and $0 < d < 1$. Thus, if *S* is the current asset price, the price in the next step will be either by *Su* or *Sd*. The up and down factors depend on the volatility of the underlying and Δt. That is

$$u = e^{\sigma\sqrt{\Delta t}} \tag{9}$$

$$d = e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u} \tag{10}$$

$$\Delta t = \frac{T}{n} \tag{11}$$

where *T* is the time to maturity, *n* is the number of steps of the binomial tree. The larger the *n*, the more accurate the calculation. The B-S model appears when $n \to \infty$. Then, the asset price at each node can be calculated using this formula

$$S(n) = S(0) * u^{Nu-Nd} \tag{12}$$

where *S(0)* represents the initial underlying price, *u* is the up factor, *d* is the down factor, *Nu* is the number of up steps and *Nd* is the number of down steps.

The next process is to get the option prices at expiration, the final node. For call options, the formula is

$$MAX[(S(T)-K),0] \tag{13}$$

and for put options, the formula is

$$MAX\big[\big(K-S(T)\big),0\big] \tag{14}$$

where the strike price is *K*, and the underlying's price at expiration is *S(T)*.

The final step is to work backward and get the option value at the first node. Although the world today is not a risk-neutral world, risk-neutral valuation, which assumes the expected return does not increase to compensate for increased risk, will give accurate option prices [11]. The pricing formula is

$$C = e^{r\Delta T}[fu \times p + fd \times (1-p)] \tag{15}$$

$$p = \frac{e^{r\Delta T}-d}{u-d}, 1-p = \frac{u-e^{r\Delta T}}{u-d} \tag{16}$$

where $1-p$ is the probability that the price of the underlying drops down, p is the probability that the price of the underlying goes up, *fu* represents the option value at S_u, and f_d represents the option value at S_d.

2.2.3 Calibration

The goal of this paper is to analyze the effectiveness of the pricing models, so getting the SSE is important. The difference between the theoretical price and the market price of the options can be calculated to compare the accuracy. The formula for the SSE is

$$\sum_{i=1}^{M} \frac{(Pi(a)-P^m i)^2}{P^m i} \tag{17}$$

where *Pi(a)* is the theoretical prices of the options, and $P^m i$ is the market prices of the options that can be found on Yahoo! finance.

Next, calibration is necessary to get optimal volatility from the SSEs. Inverse regression was used for calibration. Linear regression models the relationship between the independent and dependent variables. Independent variables are the ones been manipulated in experiments, and dependent variables are the ones affected by changes in the independent variable. Linear regression is a process of getting the unknown dependent variable based on the independent variables. Calibration is a reverse of linear regression, so it predicts the independent variable based on the observed dependent variable. When referring to model calibration, the model's parameters are acted as independent variables [14, 15]. In this case, the independent variable is the volatility, σ, and the dependent variable is the theoretical option prices. The author will be using the Solver function in the Excel spreadsheet to perform calibration. In addition, it is known that the model results in smaller SSE is more effective [9].

2.3 2.3 Hypothesis

Based on the characteristics of the models described in Section 2.2, it is reasonable to hypothesize that the B-S model is better for the European-style index options because the model assumes that the option is European, meaning it can only be exercised on the expiration day. It is also reasonable to suppose that the BT model is a better method for American equity options with no dividends because the BT model is in discrete time, which matches the early exercise feature of American options.

3. Results

3.1 Theoretical Prices

This paper uses the B-S model and the BT model to derive the theoretical option values. For calibration, later in the process, the author randomly assumed the annual volatility of both types of options, σ, to be 0.1.

3.1.1 Black-Scholes Prices

To calculate the B-S price, the five inputs described in Section 2.1 are required. Using the assumed volatility and all data in Section 2.1, the option prices can be calculated by plugging the call option data into Equation (4) and put option data into Equation (5). When plugging in the data it is important to change the time to maturity from daily to annually, which means dividing the number of days by the total number of trading days in a year. The BRK-B options have the time to maturity, *T*, as $\frac{18}{251}$, which is approximately 0.0714. It is also crucial to change percentages to decimal forms. For example, an interest rate of 1.5% becomes 0.015. The B-S prices for BRK-B options are shown below.

Table 3. B-S price of BRK-B options

Calls				
K	S	N(d1)	N(d2)	B-S price
260	273.85	0.9770	0.9755	14.1919
265	273.85	0.9002	0.8954	9.4834
270	273.85	0.7201	0.7111	5.4254
275	273.85	0.4588	0.4483	2.5164
280	273.85	0.2184	0.2106	0.9034
Puts				
K	S	N(-d1)	N(-d2)	B-S price
260	273.85	0.0230	0.0245	0.0635
265	273.85	0.0998	0.1046	0.3496
270	273.85	0.2799	0.2889	1.2862
275	273.85	0.5412	0.5517	3.3720
280	273.85	0.7816	0.7894	6.7536

The ^XEO options have a time to maturity, *T*, of 0.4701. and the B-S prices for BRK-B options are shown below.

Table 4. B-S price of ^XEO options

Calls				
K	S	N(d1)	N(d2)	B-S price
1700	1725.61	0.6388	0.6128	67.8391
1720	1725.61	0.5732	0.5462	56.3280
1740	1725.61	0.5064	0.4790	46.1483
1760	1725.61	0.4401	0.4132	37.2919
1780	1725.61	0.3762	0.3505	29.7149
Puts				
K	S	N(-d1)	N(-d2)	B-S price
1700	1725.61	0.3612	0.3872	30.2832
1720	1725.61	0.4268	0.4538	38.6316
1740	1725.61	0.4936	0.5210	48.3113
1760	1725.61	0.5599	0.5868	59.3144
1780	1725.61	0.6238	0.6495	71.5968

The process was conducted through an Excel spreadsheet. Getting the values for *N(x)* means calculating the normal distribution of *x* with mean 0 and variance 1 in cumulative form. The B-S prices will be used for calibration, which means the price will change later in the process due to calibrated volatility.

3.1.2 Binomial Prices

Same as the B-S pricing, in Binomial pricing, the author assumed the volatility to be 0.1. Equations (9) and (10) indicate that the up factor and down factor of options with the same underlying asset, in this case, is the same. For instance, the BRK-B call and put options will have the same *u* and *d*. Consequently, the probability of the price moving up, *p*, and the probability of the price moving down, *1 – p*, are the same for options with the same underlying. Through the BT model, the prices of the two options are shown below.

Table 5. BT prices of BRK-B options

Calls	
K	BT price
260	14.1848
265	9.4887
270	5.4305
275	2.5356
280	0.9221
Puts	
K	BT price
260	0.0564
265	0.3550
270	1.2913
275	3.3911
280	6.7722

Note: The author used Equations (9 – 11) to get *u* equals 1.0064, *d* equals 0.9938, and Δt equals 0.0040 and used Equation (16) to get *p* equals 0.4984 and *1 – p* equals 0.5016.

Table 6. BT prices of ^XEO options

Calls	
K	BT price
1700	67.9430
1720	56.2410
1740	46.3408
1760	37.3523
1780	29.7221
Puts	
K	BT price
1700	30.3872
1720	38.5445
1740	48.5038
1760	59.3748
1780	71.6041

Note: The author used Equations (9 – 11) to get *u* equals 1.0091, *d* equals 0.9912, and Δt equals 0.0080 and used Equation (16) to get *p* equals 0.4978 and *1 – p* equals 0.5022.

3.2 Calibration

The steps of model calibration are as follows: first, getting the SSE by using Equation (17); second, using the Solver Add-in function to minimize the SSE by changing the volatility. There will be two results at the end: new optimal volatility and the SSE. For this paper, only the SSE is important for the research. The results derived are shown below.

Table 7. SSEs of BRK-B options

	B-S model	BT model
Calls	0.0377	0.0327
Puts	0.0463	0.0477
Overall	1.5703	1.6015

Table 8. SSEs of ^XEO options

	B-S model	BT model
Calls	591.7479	591.6151
Puts	855.8841	897.9412
Overall	2391.4191	2390.989

3.3 Comparison

Table 5 and Table 6 illustrate that the results are against the author's hypothesis. The B-S model derived a smaller SSE for BRK-B options, a representative of American equity options with no dividends. And the BT model derived a smaller SSE for ^XEO options, a representative of European-style index options. Another observation from the result is that the B-S model derives smaller SSE when pricing put options, and the BT model derives smaller SSE when pricing call options.

4. Discussion

The result of the case study differs from the author's hypothesis made in Section 2.3. The author supposed that the B-S model is better for European-style index options, but the result of the case study suggests that the BT model is better for European-style index options. The author also hypothesized that the BT model is better for American equity options, while the case study suggests that the B-S model is better.

The difference between the result and the author's hypothesis on American equity options can be best explained by the original definition of the B-S model and Merton's view on rational investors. Although the B-S assumption assumed every option is European, the B-S model was developed for stock options, not index options. Nevertheless, Merton suggested that American options can be valued through the B-S model because, usually, the value of options at expiration is greater than the early exercised option value. He proposed that rational investors would trade on the expiration date. So, the B-S model can also apply to American options [12]. The research shows that the B-S model is a better model for pricing American equity options.

However, the reason for the difference between the hypothesis and the result on European-style index options is not clear. One possible difference may be because the option's underlying asset is the S&P 100 index, not a European index. Despite trading in European style, the option itself is based on an American index.

Another finding of this research is that the put options tend to obtain higher SSEs and higher volatility than call options regardless of the model used to calibrate or the style of the option. In Table 5 and Table 6, the result shows that the SSEs and optimal volatilities for put options are always greater than that of call options. This circumstance can be best explained by volatility skew. Various option pricing models assumed that the implied volatility of an option is constant. However, beginning in the 1980s, researchers found that volatilities were skewed [5], and investors were willing to pay more for put options because they wanted downside protection. Put options were thus assigned greater implied volatility as people trade put options to hedge risks in an overall bullish market. Generally, put option prices are higher than call option prices because higher volatility means a higher price. But there are exceptions. For example, in this paper, although BRK-B's put options derive higher implied volatility, its call option prices are higher than put option prices.

5. Conclusion

This paper is a comparative study of the B-S model and the BT model by using these models to price European-style index options and American equity options with no dividends. This paper aims to gather real-world data from Yahoo! Finance, compare the theoretical price with the market price, then determine which model is more effective for each of the two options. The first step of the research was gathering data. Second, calculating the theoretical prices by plugging the data into the two different pricing models. Third, calibrating the optimal volatility by minimizing the SSE.

Through this research, the author found that American equity options derive less SSE when using the Black Scholes model, and European-style index options derive less SSE when using the BT model. So, the B-S model is more effective when pricing American-style equity options, and the BT model is more effective when pricing European-style index options. The author believes that the B-S model fits the American equity option because although the model was initially developed for European equity options, it can apply to American equity options. Moreover, the author proposed that the discrete-time BT model matches the European-style index option because the underlying index is an American index.

This paper can serve as a reference to help later researchers decide the pricing model they will use for European-style index options and American equity options with no dividends. The paper is an application of the formulas and models in a real-life scenario. The author believes that there will be more researchers conducting case studies that solidify the effectiveness of the models.

There are some limitations to this research. First, the sample size is small. This paper focused on two types of options but only priced two specific options. The first one was the European-style S&P 100 index options, serving as a representation of European-style index options. The second was BRK-B (Berkshire Hathaway)'s options, representing American equity options. Many other options are European-style index options and American equity options, such as the FTSE 100 index options and Amazon.Com Inc.'s options. A larger sample size would yield a more accurate result. Moreover, when pricing using the BT model, the number of steps was not enough for more precise pricing. When pricing S&P 100 index options, the steps were limited to 59 to expedite data processing.

While these factors may lead to slight inaccuracy in the result and conclusion, expanding the sample size and adapting Python or other coding methods will have a more accurate outcome.

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