Option pricing and risk hedging for Apple

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Abstract. The Black Sholes Merton (BSM) model is one of the fundamental stochastics models in quantitative finance and the Merton Jump diffusion (MJ) model. This paper examines how BSM, and MJ behave on the European pricing based on 10 options chosen for Apple Inc, with BSM using RRS, SSE, and Historical Volatility, and MJ using SSE as calibration methods. Then delta-neutral hedging strategy is performed using the BSM on the historical data collected from the concessive 10 days. The BSM with RRS and SSE when pricing should be preferred, and the results are similar. The MJ and the BSM using Historical Volatility, however, do not work well when pricing. The delta-neutral hedging strategy is not ideal in this case, since it results in lower profits. The result possesses valuable insights for quantitative finance that calibration methods can significantly influence the accuracy of pricing, and the hedging method can limit the maximum profit.

Keywords: BSM; MJ; RSS; SSE.

1. Introduction

A wide range of calibration methods for different pricing stochastic models has been discussed and explored for a long time. The intrinsic properties of pricing models like the BSM and the MJ with varying methods of calibration could influence the accuracy, providing strategies for pricing the European options. The appropriate calibration strategy allows us to conduct accurate predictions and get better performance in hedging.

Regarding this interesting financial field, numerous investigations have been done. For example, the multiscale correction was used for the BSM by Kim, Lee, Zhu, and Yu [1]. The minimization of the sum of squared residuals of the implied volatility across all strikes was applied in calibration. The local volatility was calibrated using the Tikhonov regularization on ill-posed nonlinear inverse problems for the BSM by Crépey [2]. For the Merton Jump diffusion mode, the option pricing using Nonparametric calibration was studied by Rama, Cont and Peter, using numeric methods including gradient optimization [3]. Peter and Ekaterina calibrated the MJ by numerically minimizing the squared norms of error between prices from the model and the real market [4]. Hedging, on the other hand, is an investment strategy that could help to manage risk. Combining option pricing and hedging is also an interesting topic. For example, Robert, Myron, and Gladstein explored investment strategies for Alternative put option portfolios [5]. Trennepohl, Booth, and Tehranian conducted hedging strategies with insurance components [6]. The relation between the BSM delta and expected movement in implied volatility for an option on a stock index is shown by Hull and White [7].

With these different numeric approaches discussing the calibration methods for the BSM and MJ, different approaches applied to the same data set are rarely explored. To compare the different methods calibration methods including RRS, SSE, and Historical Volatility using the BSM for pricing, data from Apple Inc was chosen, for its pioneering performance in the technology field. Then different models, BSM and MJ, using the same calibration method SSE were discussed in this paper, to show the intrinsic properties of the model that can influence the accuracy of pricing. Delta-neutral hedging strategy is chosen to apply to the BSM model. The results of this study suggest that RRS and SSE calibration methods have better performance in pricing, and the delta-neutral strategy will bring less profit is being conducted.

2. Methods

In this project, the comparison focuses on pricing results generated by the Black Sholes Merton, model, and the Merton Jump-diffusion model. Incorporating jumps from the Poisson process, the Merton jump-diffusion model allows sudden large movements in the price of the asset. To obtain the pricing results for the Black Sholes Merton model, three calibration methods are used: RRS, SSE, and historical data generated sigma. The calibration method used for the Merton Jump diffusion model is SSE. This project also explores the delta-neutral hedging strategy using the BSM model on historical data of concessive 10 days and conducts a comparison of the profit with and without hedging strategy. Detailed empirical methods are shown below.

2.1 BSM model

Black Sholes Merton model is a powerful stochastic differential equation that performs well in pricing European options. For European options, both put and call options can be derived from BSM model [8]. Detailed BSM model is shown below.

$$c(S(t), \mathrm{K}, \mathrm{T}, \sigma, \mathrm{r}) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{1}$$

$$p(S(t), \mathrm{K}, \mathrm{T}, \sigma, \mathrm{r}) = Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1) \tag{2}$$

$$d_1 = \frac{\ln\left(\frac{S(t)}{K} + (\mathrm{T}-\mathrm{t})\left(r + \frac{\sigma^2}{2}\right)\right)}{\sigma\sqrt{\mathrm{T}-\mathrm{t}}} \tag{3}$$

$$d_2 = \frac{\ln\left(\frac{S(t)}{K} + (\mathrm{T}-\mathrm{t})\left(r - \frac{\sigma^2}{2}\right)\right)}{\sigma\sqrt{(\mathrm{T}-\mathrm{t})}} = d_1 - \sigma\sqrt{(\mathrm{T}-\mathrm{t})} \tag{4}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}} \, \mathrm{dy} \tag{5}$$

From the formulas above, c represents the European call option, and p represents the European call option. S is a function of time t, representing the stock price. K is the strike price, r is risk free rate, T is maturity, and σ is annual volatility. N(x) represents the standard normal cumulative distribution function.

2.2 Merton Jump diffusion model

Merton Jump diffusion model, cooperating results from the Black Sholes Model, also includes jumps from the Poisson process. The solution of pricing for options of the Merton Jump diffusion model [9] is shown below:

$$Price_{MJ}(S(t), \sigma, m, v, \lambda) = \sum_{k=0}^{\infty} \frac{e^{-m\lambda(T-t)}\left(m\lambda(T-t)\right)^k}{k!} Price_{BSM}(S(t), \mathrm{K}, \mathrm{T}, \sigma_k, r_k) \tag{6}$$

$$\sigma_k = \sqrt{\sigma^2 + k\frac{v^2}{T-t}} \tag{7}$$

$$r_k = r - \lambda m + \lambda + \frac{k\ln(m)}{T-t} \tag{8}$$

The price of the option in MJ model, $Price_{MJ}$, is depending on the price of option in the BSM model, $Price_{BSM}$. Exactly same as the BSM model, S is the stock price. K is the strike price, r is risk free rate, T is maturity, and σ is volatility. And corresponding to the Poisson process, m represents mean of the jump size, v represents the standard deviation of the jump size, k represents the number of jumps, and λ is the jump intensity.

2.3 Calibration Methods

For the Black Sholes Merton model, 3 calibration methods are chosen to apply in order to calibrate the volatility. The first method is Residual Sum of Squares (RSS) [10], and the formula for RSS shown below:

$$minimize(\sigma)\sum\left(\left(Price_{market} - Price_{BSM}(S(t), \mathrm{K}, \mathrm{T}, \sigma, \mathrm{r})\right)^2\right) \tag{9}$$

The second method is Sum of Squared estimate of Errors (SSE) [11], where the formula for SSE can be written as:

$$minimize(\sigma)\sum\left(\frac{\left(Price_{market}-Price_{BSM}(S(t),\mathrm{K},\mathrm{T},\sigma,\mathrm{r})\right)^2}{Price_{market}}\right) \tag{10}$$

Both methods above calibrate σ through minimize the sum of square errors between market price $Price_{market}$ and estimated price $Price_{market}$ from the BSM model. In this project, for the BSM model, there are total 10 options used for calibration. And the first five options are call options, last five options are put options, volatility is then calculated through minimize the sum of functions. Compared to RSS, SSE method further divided the price of the option from the market, allowing the formula to minimize the percentage error. Also, the SSE method for the Merton Jump diffusion model can be written as:

$$minimize(\sigma, m, v, \lambda)\sum\left(\frac{\left(Price_{market}-Price_{MJ}(S(t),\sigma,m,v,\lambda)\right)^2}{Price_{market}}\right) \tag{11}$$

Where four parameters σ, m, v, λ from the MJ model are calibrated. The third method for calibration used in this project for the BSM model is Historical Volatility method, using historical data of stock to calculate volatility. Derived from log-normal distribution identity of the BSM model, it can be shown that the:

$$\text{daily volatility} = standard\ deviation(\ln\left(\frac{S_i}{S_{i-1}}\right)), \tag{12}$$

where S_i represents the stock price at day i . The annual volatility needed for the BSM model, with τ representing the number of trading days in a year can be then shown to be:

$$\text{Annual volatility} = \text{daily volatility} \times \sqrt{\tau} \tag{13}$$

2.4 Hedging strategy on the BSM model

With historical stock price data and the BSM model, delta-neutral hedging strategy can be conducted to manage risk and results can be calculated of loss and profit with or without this strategy [12]. The historical stock price is collected from Jul. 1st, 2022, to Jul. 15th, 2022, and total 10 trading days. Also, the strike price with maturity time of the option is used in the delta-neutral hedging. First, start with day 1's data, estimated price from the BSM model at day 1, $Price_{BSM}(1)$, equals to $X(1)$. $\Delta_n = N(d_1(n))$, meaning Δ , at day n, equals to the standard normal cumulative distribution of d_1 from the BSM model, taking stock price, strike price, current time t, calibrated volatility through 10 options using SSE. The portfolio value, at day 2, is shown below to be

$$X(2) = X(1) + \Delta_1 \times [\mathrm{S}(2) - \mathrm{S}(1)] \tag{14}$$

Thus, the portfolio value at day n, where $1 \leq n \leq 10$ can be formulated to be:

$$X(n) = X(n-1) + \Delta_{\mathrm{n-1}} \times [\mathrm{S}(n) - \mathrm{S}(\mathrm{n}-1)] \tag{15}$$

Loss with hedging is the difference below:

$$Price_{BSM}(n_{end}) - X(n_{end}), \tag{16}$$

where n_{end} represents the last day of the stock. And Loss without hedging is the difference as:

$$Price_{BSM}(n_{end}) - Price_{BSM}(1) \tag{17}$$

3. Data

All data used in this project are collected from Yahoo Finance (https://finance.yahoo.com). Historical data for Hedging and calibration sigma are ten continuous trading days from Jul. 1st, 2022, to Jul. 15th, 2022 from Apple INC. The reason for choosing this company is that acting as the world's biggest company by market capitalization, it plays significant role the in the technology field. The stock price is selected to be the mean value of the stock in the present day. Open represents the stock price at opening, and the High represent the highest stock price of the day. Low represents the lowest price of the day and mean represents the average of High and Low.

Table 1. Historical stock price of Apple

Date	Open	High	Mean	Low
7/15/2022	149.78	150.86	149.53	148.20
7/14/2022	144.08	148.95	146.10	143.25
7/13/2022	142.99	146.45	144.29	142.12
7/12/2022	145.76	148.45	146.75	145.05
7/11/2022	145.67	146.64	145.21	143.78
7/8/2022	145.26	147.55	146.28	145.00
7/7/2022	143.29	146.55	144.92	143.28
7/6/2022	141.35	144.12	142.60	141.08
7/5/2022	137.77	141.61	139.27	136.93
7/1/2022	136.04	139.04	137.35	135.66

For calibration methods SSE and RSS, 10 options are chosen from Jul. 15th, 2022. First 5 options are European call option, and last 5 options are European put option. These options are chosen with consideration of relative low volatility, corresponding to the volatility smile of the position at the money. The market value, Mid, of the option is averaged through the Bid and Ask price. The stock price collocated for Jul. 15th, 2022 is 150.17.

Table 2. 10 options for pricing

Contract Name	Strike	Bid	Mid	Ask	Implied Volatility	Option Type
AAPL220722C00148000	148	3.05	3.4	3.75	30.57%	Call
AAPL220722C00149000	149	2.79	2.945	3.1	29.91%	Call
AAPL220722C00150000	150	2.32	2.385	2.45	28.52%	Call
AAPL220722C00152500	152.5	1.15	1.205	1.26	26.73%	Call
AAPL220722C00155000	155	0.52	0.535	0.55	25.59%	Call
AAPL220722P00146000	146	0.87	1.025	1.18	34.01%	Put
AAPL220722P00147000	147	1.16	1.205	1.25	30.76%	Put
AAPL220722P00148000	148	1.32	1.43	1.54	30.05%	Put
AAPL220722P00149000	149	1.7	1.795	1.89	29.42%	Put
AAPL220722P00150000	150	2.2	2.3	2.4	29.96%	Put

At last, one call option and one put option is used for pricing, and comparison over multiple calibration methods on Jul. 15th, 2022

Table 3. Options for pricing and hedging

Contract Name	Strike	Bid	Mid	Ask	Option
AAPL220722C00145000	145	5.1	5.55	6	call
AAPL220722P00155000	155	5.25	5.25	6	put

4. Results and Discussion

4.1 Results

Based on data of 10 chosen options on Jul. 15th, 2022, for mentioned before, for the BSM model, three calibrated σ's are shown below. Then the call option and put option for pricing with contract name AAPL220722C00145000 are used for pricing with 's produced from different calibration methods. Err below represents the absolute value of the difference between the result from the chosen model and the actual market price.

Table 4. Calibration parameter and accuracy of pricing

	σ	Err of call option using BSM	Err of put option using BSM
RRS	0.2795	0.257	0.178
SSE	0.2791	0.254	0.181
Historical Volatility	0.0456	0.297	0.884

It can be directly seen that σ calibrated from RRS and SSE are close to each other. And the pricing results also have similar properties, so RRS and SSE might not influence the pricing significantly. On the other hand, σ using the historical volatility method is much smaller than the other two methods, and the pricing results are worse than the others. So σ calibrated from historical volatility should not be preferred in this set of data when pricing. And it is obvious that using the BSM model to price the call option chosen holds better results than the price of the put option chosen, in all three methods.

Results for pricing exactly two same options with the same 10 options chosen from Jul. 15th, 2022 using the MJ model would be different. Instead of just the calibration parameter of volatility, the MJ model requires to calibration of four parameters: σ, m, v, λ .The results for calibrated parameters and Err of the MJ model are shown below:

Table 5. Calibration parameters and accuracy of pricing

σ	λ	m	v
0.478	0.233	-4.05e-08	1.219
	Err of call option using MJ	Err of put option Err of call option using MJ	
SSE	0.579	0.343	

The distribution of chosen options can be shown in the tables above, that the mean is close to zero, with the variance being 1.219, which is relatively large. Jump intensity, which is the average jump times per unit time, is 0.233, relatively small. It can be inferred that jumps are not common in the distribution of chosen options. And pricing results for both call option and put option using MJ behave worse than that of using BSM with RRS and SSE calibration methods. Using the SSE method, the pricing for the chosen put option behaves better than that of the chosen call option, which is the opposite of the results of the BSM model. Overall, the SSE method from the BSM model should be preferred when pricing the call option, and RRS should be preferred when pricing the put option. Results of delta-neutral hedging strategy performed on concessive 10 trading days of data, from Jul. 1st, 2022, to Jul. 15th, 2022 using the BSM model is shown below:

Table 6. Profit with/without hedging

Profit with delta-neutral hedging	Profit without delta-neutral hedging
0.004	3.055

From the table shown above, more profit is achieved when choosing not to perform the delta-neutral hedging strategy. In both cases, the profit is positive. This implies that delta-neutral hedging could constrain risk, and also confine the maximum profit gained by hedging, as shown above.

4.2 Discussion

As the results are shown above, for this specific set of data, in general, the BSM model should be preferred over the MJ model when pricing. The behavior of the MJ model might be caused by the distribution of 10 options chosen for calibration in the MJ model, where the jump frequency is not large enough. Also, the mean of the distribution is close to zero, with a relatively huge variance. This could potentially influence the calibration and make the calibration result from SSE less accurate.

For the RRS and SSE calibration methods used for pricing on the BSM, the calibrated parameter and pricing results agree with each other, which is not surprising. This is because the only difference between RRS and SSE is that SSE divided the actual market price when summing up the square error. And since the results are close to each other in the RRS and SSE calibration methods, SSE is used as the calibration method for the MJ model. The volatility for the BSM model generated by historical data, which is much lower than the other two methods, leads to a much higher error of pricing results, which could be caused by not enough days chosen for this calibration.

The delta-neutral hedging strategy is not successful in this set of data. Delta-neutral hedging minimizes the price movements of the options in relation to the underlying asset by having the overall delta at zero. This action also reduces the according maximum profit than can be achieved.

5. Conclusion

In this project, data from Apple Inc. provides the comparison between the pricing results derived from the Black Sholes Merton model and from the Merton Jump diffusion models, and the delta-neutral hedging strategy is being applied to historical data. The innovative method that no other paper did before is to apply different calibration methods to the Black Sholes Merton model and compare the accuracy. Based on the selection of five call options and five put options from Jul. 15th, 2022, different calibrated volatilities using RRS, SSE, and Historical Volatilities lead to different results of error when pricing with another call and put option, and the conclusion is that RRS and SSE calibration methods should be preferred, where RRS generates a similar result as SSE. Then SSE method for calibration is used on the Merton Jump diffusion model, and four parameters are calibrated through the same 10 options. The Black Sholes Merton model with RRS or SSE should be considered for pricing for the error is much smaller than that of the Merton Jump diffusion model.

At last, the delta-neutral hedging is used on the concessive 10 trading days from Jul. 1st, 2022, to Jul. 15th of Apple Inc., and it has been shown that more profit could be earned without performing the hedging strategy, so hedging strategy is not appropriate in this case.

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