Comparative Analysis of Portfolio Optimization Between Markowitz Model And Single Index Model

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Abstract. The key factors which investors consider when investing are the risks and returns of the assets. Moreover, they may determine efficient investment portfolios using some financial models. The goal of this paper is to compare the various optimal portfolios constructed using the Markowitz Model and the Index Model and give possible explanations for the results. The data selected for this paper are stocks from ten companies, which are Adobe Inc., International Machines Corporation, SAP SE, Bank of America Corporation, Citigroup Inc., Wells Fargo & Company, The Travelers Companies, Southwest Airline Corporation, Alaska Air Group Inc., and Hawaiian Holdings Inc. The companies are selected from the technology sector, financial services sector, and the industrial sector of the economy in order to simulate real investment decisions. In specific, twenty years of total daily return data from the companies and the S&P 500 index are collected. The statistical analysis shows that the expected return of the optimal portfolio calculated by the Single Index Model is always higher than that of the Markowitz Model. The assumption, that all stocks are independent of each other, made by the Single Index Model may provide a partial explanation for the difference in the expected returns. Nonetheless, the specific effect of the difference is yet to be examined.

Keywords: Markowitz Model; Single Index Model; Portfolio optimization.

1. Introduction

1.1 Background

The Markowitz Model, put forward by Markowitz in 1952, is the first investment theory that suggested asset correlation and diversification, that considering assets as a combination results in a different risk level than considering the assets separately. Another major investment model is the Single Index Model. Rather than measuring correlations between the assets, it evaluates the relationship between the asset and the market. Currently, they are the major models that are widely used by investors to construct their optimal portfolios. Previous research in this field had been both theoretical and practical. Some research examined the history of the formations of the models and explained the logic behind the formulas used in the models. The practical research may be divided into two groups: single model analysis and comparative analysis. Some research constructed efficient portfolios using one model, and others constructed efficient portfolios using both models and compared the results. Much comparative analysis research, though, focused on assets from the same sector in the economy, such as the technology sector, the healthcare sector, etc. Moreover, most other comparative analysis research focused on the national stock market, such as the market in India, Portugal, etc. Thus, there is not so much research on investment portfolios constructed from different sectors of the economy in the global stock market, which is a common practice for investors in real life. Therefore, it is crucial to analyze the optimal portfolio structured with 10 random selected stocks by the Markowitz Model and the Index Model in different settings, which provides a more applicable reference for investors in the global stock market.

1.2 Related Research

To test the efficiency and effectiveness of the two major investment models in realistic investment, Qu et al. included 20-year real data of 10 stocks, S&P 500, and 1-month federal funds rate to construct two optimal portfolios under 5 real-world constraints through these two models separately. By comparing two portfolio's overall returns and risk measurements like standard deviation and sharp ratio, the final result indicates that under the first three constraints -'Free', FINRA, and Arbitrary'Box'- Index Model gives a better prediction of the maximum Sharp ratio portfolio but Markowitz Model have a higher capability to forecast the minimum variance portfolio, while under Typical U.S. Open-Ended Mutual Fund and Exclusion of Index Assets constraints Index Model performs better in both portfolio. However, the calculation reveals that the stated relative advantages are far less significant to decisively testify to the superiority of these two models under certain investment conditions [1]. Susanti et al. derived a conclusion that the Markowitz Model is superior at forming optimal portfolios to the Singel Index Model by comparing portfolios constructed with these two models by processing the data of companies listed in LQ Index 45 during the Covid-19 pandemic through the Microsoft Excel application program. The treated data manifests that the expected return of the Markowitz Model is comparatively higher and there is a proportional association between the expected total return and the risk of portfolio[2]. Cohen and Pogue examined the comparison between the optimal portfolio in Indian structured by the Markowitz Model and Index Model by first aggregating weekly data of all NSE NIFTY 50 Index from 14 September 2016 to 15 September 2017 to annual data and then calculating the following risk and expected returns. The research concludes that Sharpe's Index Model has higher operability and is more appropriate to use when constructing the optimal portfolio [3].

Putra and Dana used the purposive sampling method to investigate and compare the performance of the optimal portfolios constructed using the Markowitz Model and the Single Index Model. The research used a purposive sampling method to select 28 stocks from the Indonesia stock exchange market and concluded the average return from both models has no statistical significance, but the performance of the portfolio of the Single Index Model is slightly better compared to the one of the Markowitz Model [4]. Varghese and Joseph utilized books, websites, and expert interviews to develop a descriptive study that briefly discusses the difference between the Markowitz Model and the Index model. The research concluded that the Single Index Model is more efficient in constructing efficient portfolios than the Markowitz Model because the former requires significantly less data. However, the drawback of the single index is also evident that the model may only construct a portfolio valid for the time being [5]. Hanif et al. used the purposive sampling method to conduct a quantitative study on the stocks with the LQ-45 index in the Indonesian stock exchange market during the breakout of COVID-19. The research is intended to find a combination of sticks that can form an optimal portfolio and concluded that stocks from BBCA and BRPT with carefully considered weights may reach the goal [6].

Leung et al. studied the traditional formula of portfolio optimization estimation from the Markowitz model and discovered some disadvantages. Then, the research moved on to derive a more simple, efficient, and accurate formula to estimate optimal portfolio return that may be used in wider approaches and tested its applicability for U.S investors in the U.S stock exchange market [7]. The performance of the Markowitz model versus the Single-Index model under five different limitations was compared by Chen Zeyi et. using the 20-year historical daily total return data of 10 well-known technology firms listed on the NYSE exchange. For high-risk portfolios, performance forecasting is preferable; but, when presented with low-risk investment opportunities, investors would be better served by utilizing an index model [8]. As of yet, Markowitz's CAPM model has not been demonstrated to be accurate, and investigations carried out at various points in time on stock exchange markets in various nations have produced a wide range of contentious results. Hui-Shan Lee thinks the Markowitz model may be a significant predictor of stock prices from 2000 to 2014 in the Singapore market [9], However, certain developing economies in Europe and Asia have questioned its applicability [10,11]. Unrestricted leveraged investing in the Markowitz model contradicts real-life stock investor behavior. Frazzini and Pedersen built a model with leverage and margin constraints that vary across investors and time. The result shows that low-beta stocks are held by less leverage-constrained investors (such as private equity), while high-beta stocks are preferred by more leverage-constrained investors (such as mutual funds) [12].

1.3 Objective

The objective of this paper is to calculate and compare the optimal portfolio constructed by the Markowitz Model and the Single Index Model. After introducing important concepts and performing calculations and graphs, the paper analysis provides broader implications of the results.

2. Method and Data

2.1 Markowitz Model

The Markowitz model, also called the mean-variance model, is a portfolio optimization theory designed by Markowitz in 1952. In the theory, he suggests the concept of diversification --- investing money in certain combinations of assets, called portfolios, to minimize the risk. The Markowitz model has three assumptions: the portfolio profitability is measured by its average return; the risk of a portfolio is measured by its variance; every investor is risk-averse. Based on those assumptions, the Markowitz model computes a portfolio with the highest expected return and minimal risk. It first computes the region of efficient portfolios, called efficient frontiers. Then, it finds the most efficient portfolio by finding the intersection between the capital allocation line (a line representing the risk-and-return relationship in the capital market) and the efficient frontier. The formula of the Markowitz model is,

$$\min \delta^2(r_p) = \sum \sum w_i w_j \mathrm{cov}(r_i, r_j) \quad (1)$$

$$E(r_p) = \sum w_i r_i \quad (2)$$

where r_p represents portfolio return, r_i, r_j represent the returns of asset i and asset j, w_i, w_j represent the weights of asset 1 and asset j, δ2 (r_p) represents the variance of the portfolio, cov(r_i, r_j) represents the co-variance between asset i and asset j. In addition, in order to correctly estimate the optimal portfolio from n assets, the Markowitz model would need parameters.

2.2 Index Model

The Single Index Model is a statistical model that measures the return of assets based on systematic(market-based) factors and unique(asset-based) factors. In the model, the return of the asset forms a linear relationship with any macroeconomic or microeconomic uncertainty. The equation of stock return according to the single index model is,

$$r_j = \alpha_j + \beta_j r_i + e_j \quad (3)$$

where rj represents the return of asset j, ri represents the random return, αj represents the alpha value of asset j (when the expected return of the random stock is zero), and βj represents the easiness of ri to vary with rj, and ej represents a rando firm associated with the return of asset j. Other equations of the index model also only have two variables, the unique variable (specific to assets) and the systematic variable (specific to market index): in the expected return formula ($E[r_j] = \alpha_j + \beta_j E[r_i]$),$\alpha_j$represents unique return and $\beta_j E[r_i]$represents market return. Compared to the Markowitz model, which needs extended calculations based on numerous constraints, the index model is simpler yet more powerful to show the influencers of the market. In specific, the single index model needs parameters, which are much less than those of the Markowitz model.

2.3 Data Description

This research employs historical daily total return data during the period from May 11, 2001, to May 10, 2021, for ten stocks along with the S&P 500 index and a proxy of the risk-free federal funds rate. To significantly emphasize the distinction between the portfolio constructed with Index Model and the one constructed with Markowitz Model, these ten stocks are chosen from three separate sectors according to Yahoo! finance. Adobe Inc (ADBE), International Business Machines Corp (IBM), and SAP SE (SAP) are from the technology sector. Bank of American Corp (BAC), Citigroup Inc (C), and Wells Fargo & Co (WFC) are from the financial services sector. Travelers Companies Inc (TRV), Southwest Airlines Co (LUV), Alaska Air Group Inc (ALK), and Hawaiian Holding Inc (HA) are from the industrial sector, as shown in Table 1.

Table 1. Correlation results

	SPX	**ADBE**	**IBM**	**SAP**	**BAC**	**C**	**WFC**	**TRV**	**LUV**	**ALK**	**HA**
SPX	1.00	0.66	0.65	0.65	0.60	0.70	0.56	0.60	0.54	0.46	0.39
ADBE	0.66	1.00	0.46	0.53	0.42	0.46	0.30	0.45	0.39	0.23	0.18
IBM	0.65	0.46	1.00	0.59	0.31	0.42	0.27	0.38	0.35	0.36	0.25
SAP	0.65	0.53	0.59	1.00	0.33	0.43	0.30	0.38	0.32	0.28	0.14
BAC	0.60	0.42	0.31	0.33	1.00	0.83	0.76	0.39	0.43	0.28	0.34
C	0.70	0.46	0.42	0.43	0.83	1.00	0.70	0.51	0.43	0.30	0.34
WFC	0.56	0.30	0.27	0.30	0.76	0.70	1.00	0.35	0.41	0.35	0.36
TRV	0.60	0.45	0.38	0.38	0.39	0.51	0.35	1.00	0.41	0.36	0.24
LUV	0.54	0.39	0.35	0.32	0.43	0.43	0.41	0.41	1.00	0.52	0.42
ALK	0.46	0.23	0.36	0.28	0.28	0.30	0.35	0.36	0.52	1.00	0.40
HA	0.39	0.18	0.25	0.14	0.34	0.34	0.36	0.24	0.42	0.40	1.00

From Table 2, the raw weekly data is further aggregated to monthly frequency to calculate the annualized average return, standard deviation, beta, alpha, and residual standard deviation.

Table 2. Calculation of seleted companies

	SPX	**ADBE**	**IBM**	**SAP**	**BAC**	**C**	**WFC**	**TRV**	**LUV**	**ALK**	**HA**
Annualized Average Return	7.54%	19.58%	4.75%	12.00%	11.10%	1.03%	8.89%	9.06%	9.85%	17.43%	26.87%
Annualized StDev	14.85%	31.79%	23.18%	33.91%	39.34%	42.47%	28.13%	19.96%	31.80%	37.73%	62.07%
Beta	1	1.42	1.01	1.48	1.59	2.01	1.05	0.8	1.15	1.18	1.63
Annualized Alpha	0.00%	8.85%	-2.89%	0.81%	-0.93%	-14.10%	0.95%	3.01%	1.18%	8.55%	14.58%
Annualized Residual StDev	0.00%	23.75%	17.63%	25.78%	31.40%	30.26%	23.40%	16.00%	26.82%	33.44%	57.16%

Afterwards, the MinVar and Max Sharpe of Markowitz Mode and Index Model are obtained, as shown in Table 3 and Table 4.

Table 3. The MinVar and Max Sharpe of Markowitz Model

	SPX	ADBE	IBM	SAP	BAC	C	WFC	TRV	LUV	ALK	HA	Return	StDev	Sharpe
MinVar	1.11	-0.10	0.05	-0.10	0.01	-0.23	0.14	0.20	0.00	-0.05	-0.03	0.07	0.12	0.57
Max Sharpe	0.50	0.36	-0.22	0.07	0.30	-0.66	0.21	0.36	-0.15	0.13	0.11	0.22	0.21	1.04

Table 4. The MinVar and Max Sharpe of Index Model

	SPX	**ADBE**	**IBM**	**SAP**	**BAC**	**C**	**WFC**	**TRV**	**LUV**	**ALK**	**HA**	**Return**	**StDev**	**Sharpe**
MinVar	1.44	-0.11	-0.01	-0.10	-0.09	-0.16	-0.01	0.11	-0.03	-0.02	-0.03	0.06	0.12	0.49
Max Sharpe	0.55	0.38	-0.23	0.03	-0.02	-0.38	0.04	0.29	0.04	0.19	0.11	0.20	0.22	0.90

3. Results and Discussion

As shown in Table 1, this paper examined the stock returns annually and utilized the returns for correlation analysis to produce the stock correlation matrix. The higher the correlation between the volatility of the two companies, i.e. the closer they are to 1, means that they are more correlated and their ups and downs in the stock market are more convergent; the closer they are to 0, means that they are less correlated and their performance in the stock market is more divergent. (A perfectly negative correlation (-1.0) implies that one asset's gain is proportionally matched by the other asset's loss). Therefore, we will use the low correlation coefficients for the more heterogeneous volatility for risk diversification.

It can generate the return and risk of the whole portfolio after building the correlation matrices for each of the 10 stocks, thus the research can also use the SUMPRODUCT function to get the total return and risk of the portfolio.

The portfolio with the highest return for the same level of risk and the portfolio with the lowest risk for the same projected return is obtained by computing four distinct minimal variance frontiers using solvers for the Markowitz and index models under various restrictions.

No constraints

This paper enters a dummy variable with returns equal to the upper bound of 0.5, the lower bound of -0.3, and an increment of 0.05 into the solver table, and the result is the efficient and inefficient frontier (Fig. 2). Because larger return portfolios are available with the same risk, those below the efficient frontier are not the best option. We also look at the reliability of the efficient frontier. We chose 800 portfolios with random weights to test the reliability of the minimal variance frontier. These portfolios' returns are determined by the weights, which are multiples of the predicted return of a particular firm calculated using random numbers (including shorting). Table 5 shows that the random weighted portfolios are all located right off the efficient frontier. This outcome backs up our estimates.

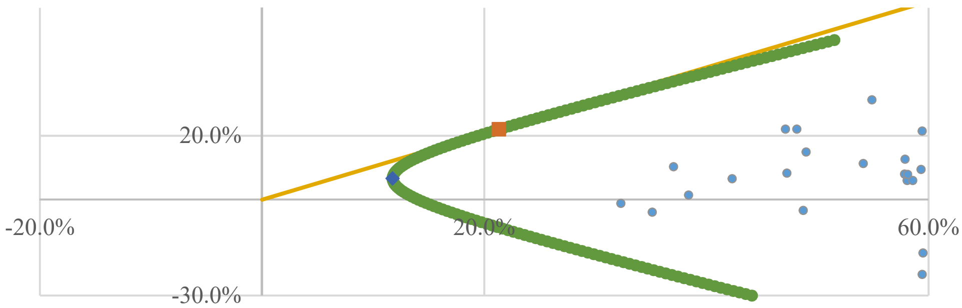

Fig. 1 Markowitz model with No constraints

"No index" problem

The second model employs Markowitz's model without the index, and defines w1= 0(The yield is borne only by the firms we collect). After eliminating the possibility of assigning weights to the index, the results of the model appear different from the first set (Fig. 2).

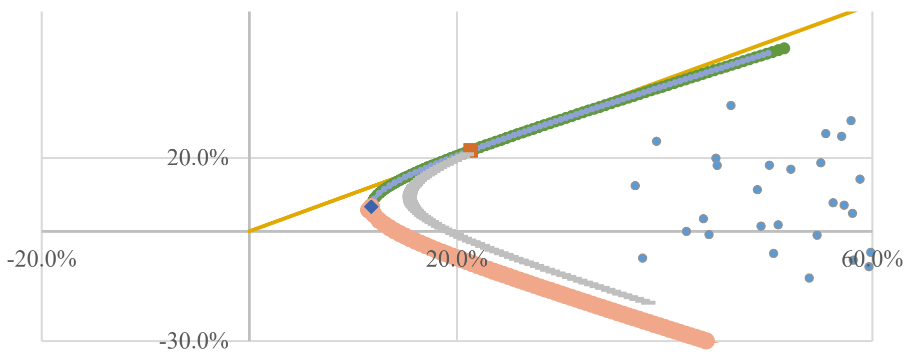

Fig. 2 Markowitz model with "no index" problem

By comparison, as shown in Table 5, it finds that after eliminating the index SPX, some points of the minimum variance frontier in the first group become inaccessible. It can be explained and accepted because some of the risks are not diversified. Thus, since the increase in the overall variance, the portfolio return as Table.5 indicates a rise from 11.2% to 14.8%.

Table 5. Returns of Markowitz model in different constraints

Condition	Return
Constraint 1: no additional constraints	11.2%
Constraint 2 : w1=0	14.8%

IM without constraints

Although the single-index model cannot analyze the relationship between different stocks, it can relate the volatility between a single stock and the market (in this case, the market volatility is SPX). Thus, we construct a new portfolio and check the difference between its weights and those of the CAPM. In the solver, we continue to specify dummy variables with returns equal to the upper 0.5, the lower -0.3, and the increment of 0.05 in Fig. 3.

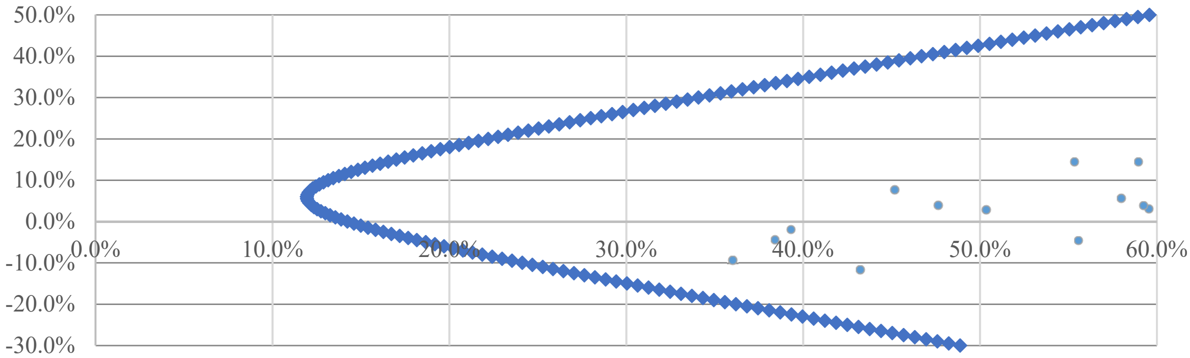

Fig. 3 Index model without constraints

IM with constraints

Similarly, we find that after eliminating the index SPX, some points of the minimum variance frontier in the first group become inaccessible, too.cc This result again confirms the idea that risks are not well-diversified without index SPX (Fig. 4).

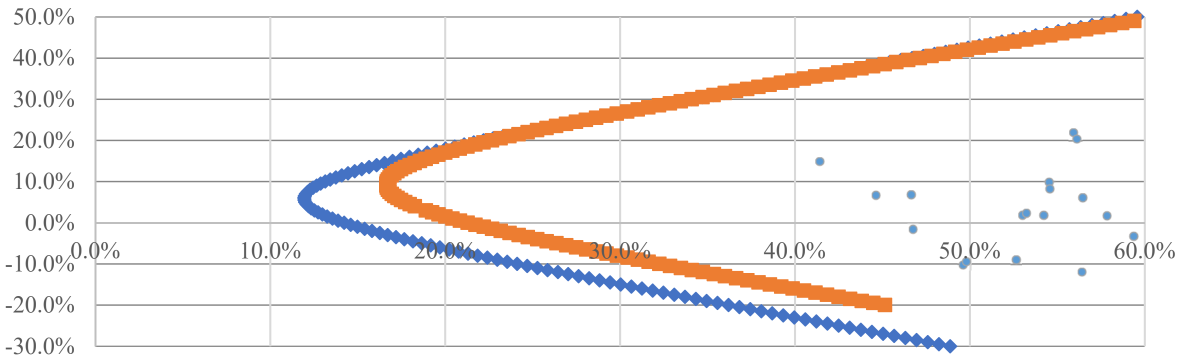

Fig. 4 Index model with constraints

Table 6. Returns of single index model in different constraints

Condition	Return
Constraint 3: no additional constraints	14.9%
Constraint 4 : w1=0	25%

By looking at the difference in their returns (Table.6), we find that the portfolio returns of the single index model are higher than those of the Markowitz model in each constraint.

4. Conclution

This paper focuses on comparing the four optimal portfolios constructed by Markowitz Model and Index Model each with two different constraints. The correlation table of stocks chosen was built first, then the total returns of 800 portfolios were calculated by the SUMPRODUCT function with different inputs according to the constraints. With no additional constraints, the expected returns were calculated with 800 random weighted portfolios, S&P500 Index, and a risk-free federal rate. With the constraint of w1=0, the total return was calculated distinctly by excluding S&P500 Index. These calculations were further transferred to minimal variance frontiers by the excel solver table. The result reveals that in either condition, the expected return of the optimal portfolio constructed by the Index Model is larger than the one constructed by Markowitz Model. We speculate this difference partially results from the Index Model's drawback of assuming all stocks are independent (i.e. correlation = 0) since the Markowitz Model will automatically consider the residual correlation of stocks included in a portfolio as building a minimal variance frontier. However, to what extent the correlation impacts the difference between Markowitz Model and Index Model hasn't been examined by this research. Thus, future research is expected to compare the total return of the optimal portfolio with different input stocks structured by the Markowitz Model and Index Model.

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