

# Pricing Options and Monte-Carlo Method a literature Review

Nier An<sup>†</sup> and Bocheng Su<sup>\*, †</sup>

University of Sheffield, S10 2TN, Sheffield, UK

\*Corresponding author: bsu3@sheffield.ac.uk

<sup>†</sup>These authors contributed equally.

**Abstract.** This literature review provides an overview of the past and present of using Monte Carlo methods to price options. From the most B-S model to combining it with a Monte Carlo method, and then from a pricing model to a method for reducing the variance of the Monte Carlo method. Furthermore, building on the solid foundation of the previous research, more recent research has focused on integrating up to several hundred dimensions and even using machine learning methods to price options. This article aims to suggest a traceable path for beginners of Monte Carlo methods, providing them with a direction for learning.

**Keywords:** Monte Carlo Method; Option Pricing; Black-Scholes Model.

## 1. Introduction

Monte Carlo has received great attention as the most popular and effective option pricing method in the market today, and it is increasingly being used by econometric financiers as computer computing power develops. It has been documented that Feynman was already using Monte Carlo in his research as early as 1930 (Metropoli, 1987), even though it did not have a precise name at the time [1]. Essentially, Monte Carlo is a method based on the law of large numbers: the integral describing the expected value of a random variable can be modelled by removing the empirical mean of an independent sample of variables. When the financial markets think of the mathematical term 'expectation', we unconsciously think of the financial derivative - options - that relate to the expectation that a stock will rise or fall.

This essay attempts to provide a rough guide to the use of Monte Carlo methods in option pricing for those who are still unfamiliar with them. The second part covers the basic idea of the Monte Carlo Method and some early research on pricing options by the MC method; the third part briefly describes the progress of Monte Carlo methods as pricing models have evolved; the fourth part focuses on recent research, and the last part discusses future directions.

## 2. Basic Idea and Early Research

### 2.1 Background: Pricing Model

Pricing models are essential to pricing options, so an introduction to the famous BS model is necessary as background to this early research. Both following papers were published in 1973, the former on the basis that if the market price is correct then the arbitrageur cannot use a portfolio with both a short and a long position to make a definite profit; the latter extended and tested the B-S model by introducing new assumptions and it also discusses the impact of dividend and redemption clauses.

Before Black and Scholes (1973) published their revolutionary paper on the B-S model, there was a lot of literature on pricing models, but they all had a similar problem: one or more unknown parameters in the model [2]. Several different pricing models have been published since this article was posted, but they all have some similar problems: they have one or more unknown parameters. Based on the basic logic that the return on a portfolio of hedged positions in equities and options should be equal to the return on a risk-free asset, the authors have assembled an option pricing formula with computable parameters in an "ideal" market. The key to this model is that, in any finite interval, the possible stock prices follow a log-normal distribution, and the variance of stock returns is constant. The option pricing expression given by the BS model:

$$\omega(x, \tau) = xN(d_1) - Ke^{-r\tau}N(d_2) \quad (1)$$

Where  $\omega(x, \tau)$  is the option price with the spot price of stock  $x$  and  $\tau$  which is the difference between the exercise date and now,  $K$  is the strike price,  $r$  is the risk-neutral rates,

$$d_1 = [\log(x/K) + (r + 1/2\sigma^2)\tau]/\sigma\sqrt{\tau}; d_2 = d_1 - \sigma\sqrt{\tau} \quad (2)$$

The authors use the model to price warrants, common stock and bonds, and the empirical study shows that buyers in the market consistently pay more than the predicted price, while sellers receive roughly the same price as predicted by the formula.

Merton discussed with Black and Scholes when they were studying the B-S model. After the B-S model was proposed, Merton made a broader exploration on its basis [3]. This article first discusses the pricing of American options not considered in the B-S model. Morton's findings are counter-intuitive, most people think that because American options are more flexible which means prices should be higher, but Merton concludes that American options should be priced only a little differently from European options. He then gives an alternative derivation of the Black-Scholes model and discusses the case where there is an interest allocation. Finally, he attempts to use it to price American options, "down-and-out" options, and callable warrants.

## 2.2 Basic Idea of Monte Carlo on Option Pricing

Boyle first attempted to link Monte Carlo methods to option pricing in 1977 [4]. His logic is as follows: If we have a distribution of a stock's final value, then the option's expected value can be obtained simply by integration. But generally, these types of integrals do not have analytical solutions, so we need to use numerical values for estimation. And the Monte Carlo method uses enough "experiments" to get the results, which met the restriction. The basic idea of Monte Carlo on option pricing is following:

In option pricing, we always need to calculate the expectation of a random variable. Imagine the future price of an underlying asset (usually a stock) is the random variable  $X$ , and then function  $g$  is transforming the stock price to the option payoff.

For a continuous random variable:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x)dx \quad (3)$$

Where  $f(x)$  is the probability density function with

$$\int_A f(x)dx = 1 \quad (4)$$

With the Strong Law of Large Number: Suppose we have a sequence of random variables ( $X_i$ ), which are mutually independent and identically distributed with the same distribution as  $X$ . Then we have the estimator of  $E(g(X))$  is given by

$$\hat{g} = \frac{1}{n} \sum_{i=1}^n g(X_i) \rightarrow E[g(X)], (n \rightarrow \infty) \quad (5)$$

Then Boyle used a Monte Carlo approach to price options where the underlying is a dividend-paying stock (the B-S model was not used because it does not consider that the stock pays dividends. But again, the final price of the stock is assumed to follow a log-normal distribution.), and its advantages and disadvantages are discussed. Overall, he pioneered the use of Monte Carlo methods to price options. Subsequent research can follow this line of thinking to price American options: using path-dependent integrals. For security with dividends: Use an iterative procedure to advance the

simulation step by step. Treating variance as a constant is not entirely reasonable since market and investor sentiment is constantly changing. We can assume that volatility should follow some probability distribution. At that time, finding out or describing what distribution it is was an important further direction.

### 2.3 Variance Reduction of Monte Carlo Method

Reducing variance is important when discussing simulations and computations with errors. This allows simulations to more quickly approximate “real” results. More specifically, if the variance is reduced by 10 times, then the number of simulations’ times will be 1/100 of that of the basic Monte Carlo method needed. Rosenberg published an article as early as 1967 discussing how to reduce the Monte Carlo method [5]. Although published earlier than the previous articles, the solution he proposed is still in use today. Three main methods for reducing variance are control variate, antithetic variate (a special form of control variate) and importance sampling. The main idea of the first two is to add a control variable that does not affect the expected value, and the last is to reduce the number of cases where the sample does not conform to the probability distribution. Each method performs well on its own. Besides, author performs a combination of these methods as these variance reduction methods are all additive and finds that importance sampling followed by control variate works best.

## 3. Efforts on Promoting Monte-Carlo

### 3.1 Background: Development of Pricing Model

After the Black-Scholes model pushed option pricing models to new heights, there has been widespread use of models that relate option prices to asset return distributions. Because we cannot determine the mean and variance of the log-normal distribution of the hypothetical price, that is, the B-S model cannot fully account for the fact that the volatility is not constant. Based on previous empirical research that BS does not perform well in FX options, Heston (1993) considers the model while taking volatility into account as a random variable [6]. Different from the second term of the BS model ( $\sigma S dz$ ), this model adds  $v(t)$  as a function of controlling volatility:

$$dS(t) = \mu S dt + \sqrt{v(t)} S dz_1(t) \quad (6)$$

$$dv(t) = k^* [\theta^* - v(t)] dy + \sigma \sqrt{v(t)} dz_2 t \quad (7)$$

Where  $k^*$  is the mean reversion,  $\theta^*$  is the Long-Run variance,  $v(t)$  is the current variance,  $z(t)$  is the Wiener process, and  $\sigma$  is the Volatility of volatility parameter.

For options prices, any type of deviation can be included, and this deviation is related to the dynamic asset price distribution. However, this does not mean that the Black-Scholes model does not perform well, as both the Black-Scholes model and the stochastic volatility model produce the same price for at-the-money options, which means that the asset return distribution has the same volatility for at-the-money options that are traded near the price. However, after considering the volatility of asset returns, and the correlation between asset prices, the stochastic model allows flexibility in pricing options on assets with stochastic volatility.

Following the Black-Scholes and stochastic volatility model, Heston and Nandi (2000) have worked on option valuation models in a further way, which is discrete-time Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models [7]. As mentioned in the previous paper, considering the correlation between the volatility of the spot price and the spot return is valid for pricing stochastic volatility models. This paper further introduces the path dependence of volatility and proposes a GARCH model. The model expresses a function of the observed path of current spot prices and historical spot prices and can capture both the correlation between volatility and spot returns and the path dependence of volatility to value options, as well as predict out-of-sample option

prices. For markets in which there are no reliable option prices in illiquid markets, it is difficult to use other contemporaneous options to obtain implied volatilities to value options. Moreover, volatility is not inherently observable and continuous-time stochastic volatility models are difficult to implement and validate, even if the models have assumed that volatility will be observed. GARCH models estimate the parameters of the formula from discrete observations, which is an easy function to calculate. However, existing GARCH models do not have closed-form solutions for option values. This paper provides a closed-form solution for European-style options in the GARCH model, setting the option price as a function of the current and lagged spot price. In contrast to the Black-Scholes model, the option value is a function of the current spot asset price and allows for multiple lags in the time series dynamics of the variance process. It also allows for a correlation between the return on the spot asset and the variance. Finally, the paper tests the model by using data from the S&P 500 index options market and shows that the valuation error (out-of-sample) of the GARCH model is smaller than that of the ad-hoc BS model.

The asset return distribution has a higher peak and two heavier and asymmetric tails (leptokurtic features) compared to the normal distribution and a "smile-shaped" implied volatility curve in the options market. This is considered by most to be a standard phenomenon. Many academics have worked on new models to modify the Black-Scholes model and incorporate the 'volatility smile' into option pricing. But these models may be difficult to obtain analytical solutions for option prices. Kou (2002) presents a proposed double exponential jump-diffusion model in which the logarithm of asset prices is assumed to follow a Brownian motion plus a compound Poisson process, so that the model is simple enough to solve a variety of option pricing problems, including call, and put options, interest rate derivatives, and path-dependent options [8]. At the same time, the article also suggests the limitations of the double exponential jump-diffusion model; the formulation appears complex, but it is a closed-form solution, so it is easy to obtain in computer programming. Also, the dependence structure that exists between asset returns is not considered by the model as it assumes independent increments. Finally, the risk-free hedging theory cannot be considered by the model because the jump model does not contain complete market information.

### 3.2 Monte-Carlo

Based on the pricing model described above, more literature on pricing using Monte Carlo methods is stated below. A large part of them is talking about how to reduce variance.

Broadie and Kaya (2006) first consider Heston's Stochastic Volatility model, then briefly reviews Euler discretization, and then propose an Exact Simulation method derived from four steps [9]. They compared the RMS error and runtime obtained from simulations using it with Euler discretization and found that Exact Simulation far outperformed Euler discretization in both aspects. Considering the Conditional Monte Carlo based on the SV model, the relationship between the pros and cons of the two methods remains. The Exact method performs better in both running speed and residuals. Further, Broadie and Kaya extended the Exact method to more models (such as the stochastic volatility with jumps model, and stochastic volatility with contemporaneous jumps model), and found that the Exact method still has better performance than Euler discretization. Overall, the Exact Method restores error convergence in SV models and can be generalized to mild, weak path-dependent pricing unbiased estimators.

Dingeç and Hörmann's (2013) study focuses on variance reduction methods for basket options and Asian options [10]. When it comes to reducing variance, it is inevitable to refer to the classic control variable method, so the author first describes the traditional method in basket options and Asian options. Then he proposed a method of dividing the profit function into two parts, introducing  $G$  as a conditioning variable, and giving a new CV estimator:

$$Y_{CV} = P_A - c(W - EW)) \quad (8)$$

Where the native simulator is PA and the control varieties  $W=(A-K)1G>K$ . By integrating the author has obtained the following closed formula:

$$\mu_w = \left( \sum_1^d \omega_i e^{\tilde{\mu}_i + \tilde{\sigma}_i^2/2} \Phi(-k + a_i) \right) - K\Phi(-k) \quad (9)$$

After optimization and testing, they found that this new method of controlling variables produced a considerable reduction in variance, far better than the traditional method.

### 3.3 Quasi-Monte-Carlo

Likewise, it is worth mentioning Quasi-Monte Carlo, a method that was found to perform well in high-latitude integration in a follow-up study. Next is its Basic idea and its application in asset pricing.

Caflisch (1998) discusses the application of Monte Carlo and Quasi-Monte-Carlo to integration [11]. In the first half of the article, the author discussed the traditional Monte Carlo method and the variance reduction method mentioned above. Starting from Chapter 5, he introduced the Quasi-Monte Carlo method and compared it with the Monte Carlo method. Caflisch stated that the Quasi-Monte Carlo method is far superior to the traditional Monte Carlo method in terms of convergence rate because the convergence rate of the traditional Monte Carlo method using N samples is:  $O(N^{-1/2})$ , but if the Quasi-Monte Carlo method is used, the convergence rate can be changed to  $O((\log N)k/N)$  which is derived based on the Koksma-Hlawkain-equality theorem. Although the authors study this method more focused on its contribution to the field of physics, the advantages of the pseudo-Monte Carlo method in integrating multiple dimensions can also be used in finance.

Boyle et al (1997) work on applying Monte Carlo methods to asset pricing [12]. Except for the inconsistency in the research direction, he is generally like the previous paper: he starts with the traditional Monte Carlo method, then introduces several variance reduction methods, then tries to apply the Quasi-Monte Carlo method to stock pricing, and finally, they Attempt to use path simulation to price American options that are generally considered im- possible to price using simulation. When they empirically tested the Quasi-Monte Carlo method, they found that simply replacing the pseudo-random number generator with a low-discrepancy sequence generator would lead to huge errors due to the exacerbation of the 'collinearity' or 'hyperplane' problem. A more complete Quasi-random number generator may be needed in the future. They finally tried to develop a pricing formula for American options. They also thought that if they wanted to find a pricing formula, they needed to find an optimal path algorithm, which provided direction for the follow-up research.

## 4. Recent Research

Liang and Xu's (2019) research focuses on pricing multi-asset options [13]. The Monte Carlo method was selected among the three mainstream pricing methods (the analytic approximation method, the fast Fourier transformation method, and the MC simulation method.). Liang and Xu first used the SV model to price multi-asset options, and then innovatively combined martingale control variate and Monte Carlo methods. After testing, this method can significantly reduce the variance. But the author has not extended this method to more forms of option pricing.

For Quasi-Monte Carlo, the most mentioned by scholars is its excellent performance in high-dimensional integration, and this piece is no exception. Todorov et al (2018) in their study use quasi monte Carlo to price options based on LATTICE SEQUENCES (Based on Fibonacci Generating Vectors and Good Generating Vectors) [14]. The article conducts an empirical study on the effectiveness of using different lattice rules and finds that it is better to use LAT2 (lattice rules based on polynomial transformation functions) for non-smooth integral functions, and LAT1 (lattice rules based on aperiodic transformation functions) for low-dimensional cases work more efficient.

Gan et al (2020) pioneered the combination of machine learning and option pricing without the use of models [15]. Gan et al experimented with using machine learning to price average options by

reviewing traditional options pricing methods and using them to create artificial option price data for training and testing machines. By using the Back Propagation (BP) neural network in Tensorflow, they built their deep learning framework and trained using the above artificial data, giving results showing that the effectiveness of deep learning is robust on the test set. Going a step further, they tried training with real option price data and found that more than half of the option price predictions had a relative error of less than 0.8%. This is an amazing result. It is worth emphasizing that in machine learning, the author does not make assumptions about the probability distribution of the price of the underlying asset. It only needs to formulate the parameters in the option pricing model and give the machine enough sample data to give produce a predicted value with minimal deviation. This scheme sidesteps the debate over assumptions in various pricing models and presents a new path. Nevertheless, it also has limitations. In this study, the author mainly studies Asian options based on geometric mean. Options with complex rules such as American options may require further improvements in models and back-testing.

## 5. Conclusion

The Monte Carlo method has played an important role in the long history of options pricing. In more and more subdivided research, many scholars have devoted themselves to using the Monte Carlo method to price specific types of options for specific pricing models. Most people's research directions are looking for ways to reduce variance and dimensionality, and some people have proposed ways to improve the running speed. But just like the last piece of literature, the author believes that machine learning will play an extremely important role in the financial field. In the 40 years of studying options pricing, scholars have proposed various pricing models based on (in detail) different assumptions. Many assumptions divergences can be avoided if options are priced using machine learning methods without model assumptions. But this is also a "chicken or egg" question, and no pricing model would be proposed without assumptions. Monte Carlo methods still play an important role in validating pricing models. But machine learning may be a better solution in terms of speed and resource usage. In general, future research on option pricing may not only be all scholars delving into better methods to reduce variance and dimensionality, but also more research should focus on algorithm optimization using machine learning to price options.

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