Portfolio Optimization under Real Constraints

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Abstract. How to balance risk and return is a question that every investor cares about. Since there is little research on portfolio management in specific industries under special constraints, this paper aims to conduct asset allocation analysis on consumer defensive, technology, financial service and healthcare industries. This paper selects ten representative companies from these industries and combine them with generalized indices and use Markowitz model and index model for analysis. Portfolio optimization is based on investor preferences and five constraints of financial market regulations. The results show that, first, disallowing shorting has the strongest binding power. Second, the S&P 500, as the best single measure of U.S. large-cap stocks, is a good choice for balancing risk and reward. These findings may be of great significance to the research on the optimal allocation of financial assets in the industry and to investors with similar requirements to make their own investment decisions.

Keywords: Constrained Portfolio Optimization; Markowitz model; Index model; Sharpe ratio.

1. Introduction

The mean-variance method proposed by Harry Markowitz in 1952, for the first time, met the conflicting goals of investors who want high profits and low risks by using mathematical statistics [1]. This model analyzes and solves the investment portfolio with a simple but rigorously formulated theory, which is applicable to common practical situations [2]. On this basis, a large number of related studies have gradually started, such as the single index model proposed by Sharpe in 1963, the capital asset pricing model proposed in 1964, and arbitrage and other models.

After carefully studying the research on investment portfolio, it is found that most of the current research focuses on the following three aspects: the expansion of the theory itself; the survey based on the entire market; and the influence of investors' personal circumstances on investment decisions. For example, Deng, Lin, and Lo used a modified particle swarm optimization to solve the cardinality-constrained Markowitz portfolio optimization problem [3]. In order to solve the situation of data uncertainty when constructing a portfolio, Bhattacharyya, Hossain and Kar proposed a fuzzy stock portfolio selection model, which maximizes the mean and skewness and minimizes the variance and cross-entropy of the portfolio [4]. A hybrid two-stage robustness approach was also proposed by Atta Mills and Anyomi [5]. Besides, Martin and Lukáš constructed an optimal portfolio based on the US stock market [6]. At the same time, some researchers focus on the impact of investors' own reasons on investment choices. For example, Basheer and Siddiqui explained investors' investment propensity from several aspects of personality factors, financial knowledge, behavioral tendencies and risk tolerance [7]. The study of Bressan, Pace, and Pelizzon demonstrate that poor investor health may also have a negative impact on portfolio selection [8].

Based on the above research, this article will conduct asset allocation on ten representative companies in several specific fields in the market and discuss the results under different constraints. The specific demonstration process is as follows. First, this paper selects the SPX 500 index and ten outstanding companies in consumer defensive, technology, financial service and healthcare industries, and obtains their monthly adjusted closing prices for the past 20 years; secondly, calculate the annualized average return, annualized standard deviation, beta coefficient, annualized alpha coefficient and annualized residual standard deviation, and construct a correlation matrix; third, use the solver table to construct the minimum variance portfolio and the maximum Sharpe ratio portfolio under the two models and different constraints, and obtain the capital allocation line and the minimum variance boundary; fourth, use excel to draw and get visual results and analyze portfolio performance in various scenarios.

This paper is constructed as follows. Section 2 presents the data used in this article. Section 3 describes the methodology used. Section 4 presents the constraints and associated results. Section 5 is the conclusion.

2. Data

The data used in this paper is derived from Yahoo Finance (https://finance.yahoo.com/). This article analyzes S&P 500 and ten companies, selected from representative companies in the fields of Technology, Finance Services, Consumer Defensive, and Health Care, namely NVDV, CSCO, INTC, GS, USB, TD CN, ALL, PG, JNJ, CL. In order to reduce the impact of data fluctuations, the monthly closing prices from November 1, 2002, to November 1, 2022, were selected. In order to construct financial portfolios, all the closing prices are transferred to returns. Finally, 241 data were collected. By processing the data, the annualized average return (AAR), annualized standard deviation (ASD), beta coefficient, annualized alpha coefficient and annualized residual standard deviation (ARSD) and the correlation coefficient matrix of each stock are obtained and is shown in Table 1.

Table 1. Descriptive statistics of the selected stocks

	SPX	NVDA	CSCO	INTC	GS	USB
AAR	6.804%	34.695%	9.471%	6.450%	11.842%	8.130%
ASD	14.785%	51.340%	26.372%	26.902%	29.434%	23.510%
$\boldsymbol{\beta}$	1.000	1.790	1.166	1.016	1.410	0.986
$\boldsymbol{\alpha}$	0.000%	22.518%	1.536%	-0.466%	2.248%	1.421%
ARSD	0.000%	43.995%	19.955%	22.313%	20.777%	18.442%
	TD CN	ALL	PG	JNJ	CL	
AAR	14.207%	10.321%	8.382%	8.107%	7.499%	
ASD	23.129%	24.675%	14.973%	14.857%	15.603%	
$\boldsymbol{\beta}$	1.051	1.061	0.449	0.551	0.485	
$\boldsymbol{\alpha}$	7.057%	3.103%	5.328%	4.354%	4.201%	
ARSD	17.132%	19.048%	13.422%	12.420%	3.859%	

Through data visualization, it can be concluded that NVDA has the highest annualized average return, while INTC has the lowest. In terms of annualized standard deviation, the S&P 500 has the smallest index and NVDA has the highest.

3. Methods

The Markowitz model, also known as the mean-variance model, is a theory proposed to meet the needs of investors. The model aims to find the best balance point between risk and investment return under ideal conditions, so that the utility of the asset portfolio can be maximized [9]. It uses mathematical and statistical techniques to maximize a portfolio's return at an investor's level of risk, or to minimize risk at an investor's expected return [10]. The Markowitz model shows that diversification ensures that unsystematic risk is eliminated by combining different assets in a portfolio, since the poor performance of some assets in the portfolio can be offset by the positive performance of other assets in the portfolio. Because the model has strong operability and wide applicability in the process of portfolio optimization, it laid the foundation of modern portfolio theory.

To find the expected return of a portfolio, the mean, variance, and correlation coefficient of each asset or security included in the set must be known. Therefore, in this paper, the set of assets' annualized average return is expressed as $\vec{\mu} = \{\mu_1, \mu_2, \dots, \mu_n\}^T$; the set of assets' annualized standard deviation is $\vec{\sigma} = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}^T$; the set of the residuals' annualized standard deviation is $\{\sigma(\varepsilon_1), \sigma(\varepsilon_2), \ldots, \sigma(\varepsilon_n)\}^T$; the matrix of assets' correlation coefficient is shown below

$$\mathrm{P} = \begin{Bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{Bmatrix} \tag{1}$$

The uncertain set of assets' weights is $\vec{w} = \{w_1, w_2, \ldots, w_n\}$; the set of assets' betas is $\vec{\beta} = \{\beta_1, \beta_2, \ldots, \beta_n\}^T$; and the auxiliary vector is $\vec{v} = \{w_1\sigma_1, w_2\sigma_2, \ldots, w_n\sigma_n\}^T$. An important premise of the Markowitz model is

$$\sum_{i=1}^{n} w_i = 1 \tag{2}$$

Besides, the formula for the return by Markowitz model (MM) is

$$r_P = \vec{w} \cdot \vec{\mu}^T \tag{3}$$

The MM portfolio standard deviation is shown as

$$\sigma_P = \sqrt{\vec{v} P \vec{v}^T} \tag{4}$$

The index model (IM) is an asset pricing model developed by William Sharpe in 1963, it is also one of the important methods used to measure the risk and return of stocks. According to the model, any economic factor related to a security can be expressed as a linear relationship with the return of the stock. By definition, each instrument has a regression, so the expression for the IM portfolio return is

$$r_P = \vec{w} \cdot \vec{\beta}^T \cdot r_M + \vec{w} \cdot \alpha_i \tag{5}$$

Where r_M represents the return to the market portfolio, in this paper it refers to S&P 500; and α_i alpha indicates the abnormal return of the stock. Besides, portfolio beta is denoted as $\beta_P = \vec{w} \cdot \vec{\beta}^T$. Then, the IM portfolio standard deviation is written as

$$\sigma_P = \sqrt{(\sigma_M \beta_P)^2 + \sum_{i=1}^{n} w_i^2 \sigma^2(\varepsilon_i)} \tag{6}$$

In addition, the Sharpe Ratio compares an investment's return to its risk, known as the standard deviation, and is an important method used to measure investment performance after risk adjustment. It is of the form:

$$Sharpe\ Ratio = \frac{r_p - R_f}{\sigma_p} \tag{7}$$

Where R_f is the risk-free rate.

4. Results

Through the calculation of the correlation coefficient matrix between these ten stocks, the correlation coefficient between SPX and GS is the largest, and the correlation coefficient between

NVDA and CL is the smallest. Besides, it can also be seen that all correlation coefficients are greater than 0, so all stocks are positively correlated (See Table 2).

Table 2. Correlation matrix

Correlations	SPX	NVDA	CSCO	INTC	GS	USB	TD	ALL	PG	JNJ	CL
SPX	1	0.515	0.654	0.559	0.708	0.620	0.672	0.636	0.443	0.549	0.459
NVDA	0.525	1	0.426	0.431	0.331	0.173	0.352	0.184	0.106	0.119	0.089
CSCO	0.645	0.426	1	0.541	0.477	0.407	0.447	0.430	0.305	0.291	0.253
INTC	0.559	0.431	0.541	1	0.401	0.325	0.403	0.361	0.169	0.340	0.155
GS	0.708	0.331	0.477	0.401	1	0.508	0.539	0.434	0.170	0.308	0.210
USB	0.620	0.173	0.407	0.325	0.508	1	0.544	0.518	0.320	0.233	0.248
TD	0.672	0.352	0.447	0.403	0.539	0.544	1	0.499	0.291	0.288	0.303
ALL	0.636	0.184	0.430	0.361	0.434	0.518	0.499	1	0.367	0.501	0.379
PG	0.443	0.106	0.305	0.169	0.170	0.320	0.291	0.367	1	0.485	0.566
JNJ	0.549	0.119	0.291	0.340	0.308	0.233	0.288	0.501	0.485	1	0.516
CL	0.459	0.089	0.253	0.155	0.210	0.248	0.303	0.379	0.566	0.516	1

Considering investor preferences and legal or industry regulations, this paper will construct investment portfolio under five additional constraints. By calculating the minimal risk frontier, minimal risk portfolio, optimal portfolio and the capital allocation line under different conditions. In the following tables, MinVar represents the minimum variance and MaxSharpe represents maximum Sharpe ratio.

Constraint 1: In the securities market, there will always be situations where investors do not have sufficient funds. As a result, the securities transaction fee cannot be reimbursed in a timely manner. When this situation arises, the investor can apply for credit from the dealer or broker to purchase the security on margin. However, due to the uncertainty of risk, investors are likely to lose more money when trading on margin than they have deposited cash in the economic account. Therefore, in order to protect both sides of margin trading and reduce risks, the Federal Reserve Board and the Financial Industry Regulatory Authority (FINRA) stipulate the proportion of the amount of margin investors purchase. Therefore, this constraint requires that the user's margin amount must not exceed 50% of the purchase number of securities, which can be expressed as follows and the related results are shown in Table 3 and 4, Figs 1 and 2:

$$\sum_{i=1}^{11} |w_i| \leq 2 \tag{8}$$

Table 3. Weights of each asset of MM under constraint 1

MM	SPX	NVDA	CSCO	INTC	GS	USB	TD CN
MinVar	48.406%	-2.717%	-3.465%	4.031%	-6.297%	4.657%	1.807%
MaxSharpe	-41.28%	20.971%	-0.038%	-8.619%	4.537%	4.382%	27.531%
	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	-12.337%	24.395%	24.894%	16.625%	6.209%	11.535%	0.538
MaxSharpe	0.090%	34.580%	44.034%	13.817%	16.226%	15.995%	1.014

Table 4. Weights of each asset of IM under constraint 1

IM	SPX	NVDA	CSCO	INTC	GS	USB	TD CN
MinVar	32.907%	-4.186%	-4.283%	-0.340%	-9.746%	0.418%	-1.780%
MaxSharpe	-49.453%	14.031%	0.033%	-0.475%	0.076%	0.476%	25.984%
	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	-1.721%	31.385%	29.828%	27.516%	5.921%	10.128%	0.585
MaxSharpe	5.335%	40.026%	35.466%	28.501%	14.133%	14.205%	0.995

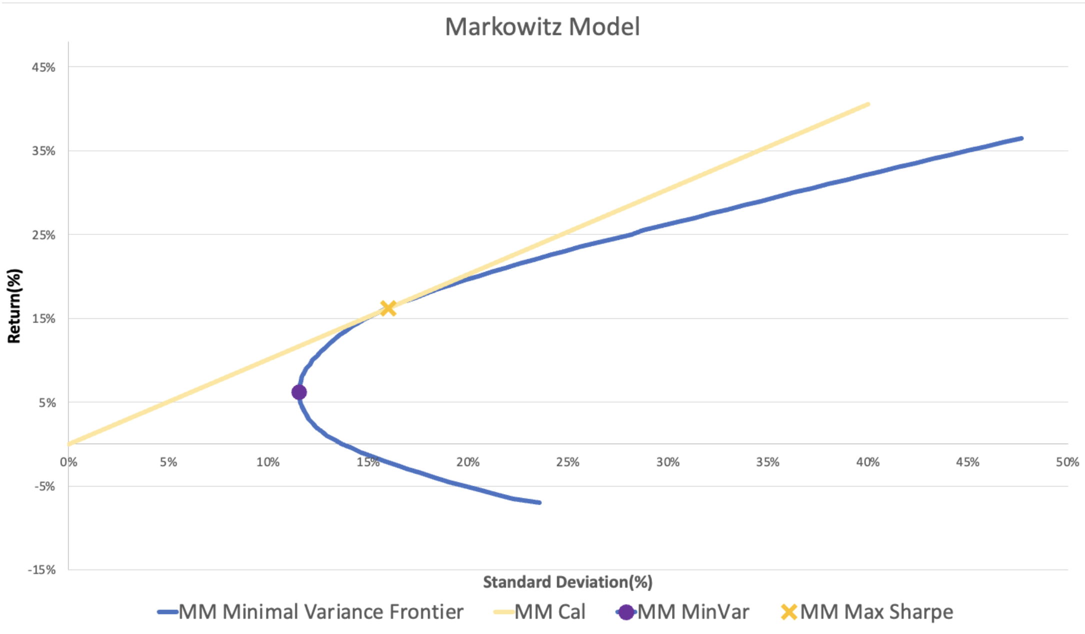

Fig. 1 Frontier, optimal portfolio and capital allocation line of MM under constraint 1

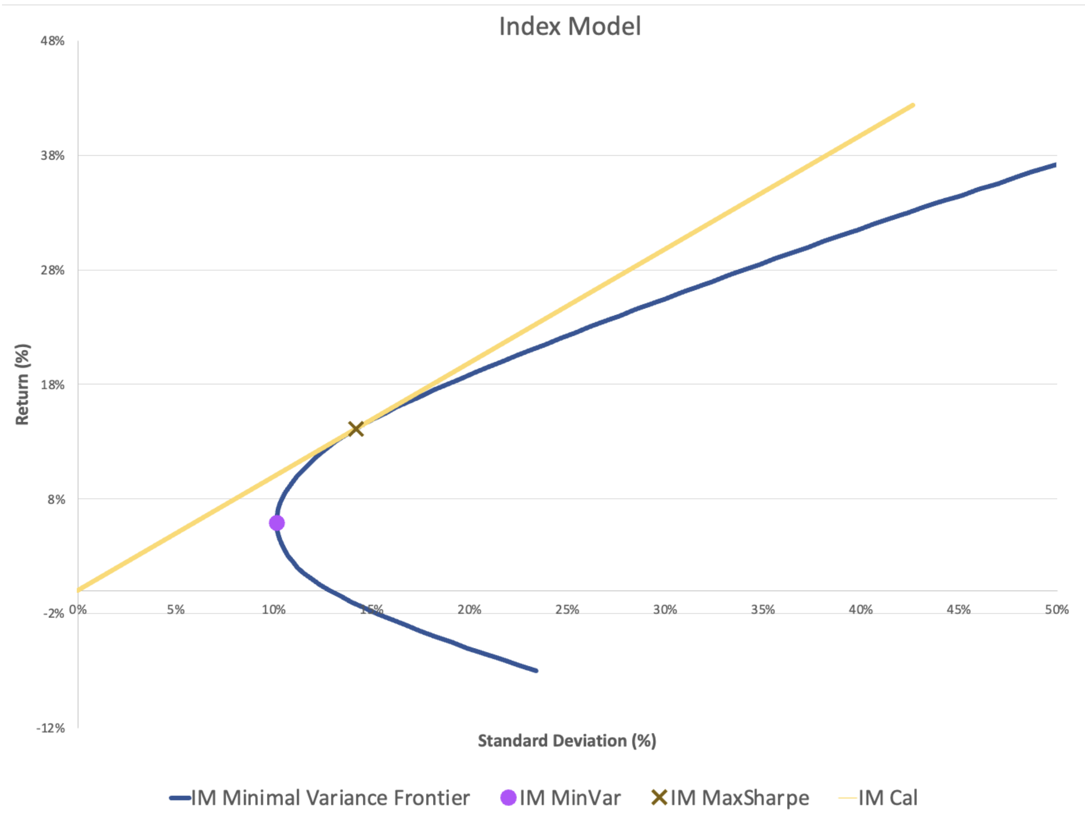

Fig. 2 Frontier, optimal portfolio and capital allocation line of IM under constraint 1

Constraint 2: Since each investor's living environment, financial literacy, and return needs are different, investment preferences will also vary greatly. It is impossible to consider the needs of every investor in this article, so this constraint can only set a possible special "box" constraint on weights, which is shown as follows and the related results are shown in Table 5 and 6, Figs 3 and 4:

$$|w_i| \leq 1, for\ \forall i \tag{9}$$

Table 5. Weights of each asset of MM under constraint 2

MM	SPX	NVDA	CSCO	INTC	GS	USB	TD CN
MinVar	48.407%	-2.717%	-3.465%	4.031%	-6.297%	4.657%	1.807%
MaxSharpe	-100.000%	28.048%	0.650%	-14.965%	13.722%	12.061%	36.281%
	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	-12.337%	24.395%	24.894%	16.625%	6.209%	11.535%	0.538
MaxSharpe	2.903%	40.594%	61.536%	19.170%	19.912%	18.564%	1.073

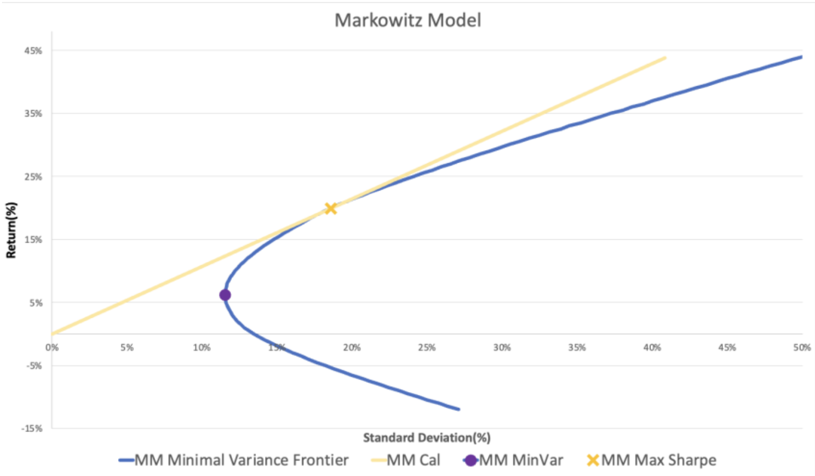

Fig. 3 Frontier, optimal portfolio and capital allocation line of MM under constraint 2

Table 6. Weights of each asset of IM under constraint 2

IM	SPX	NVDA	CSCO	INTC	GS	USB	TD CN
MinVar	32.907%	-4.186%	-4.283%	-0.340%	-9.746%	0.418%	-1.780%
MaxSharpe	-100.000%	17.367%	3.705%	-2.831%	5.212%	4.659%	35.183%
	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	-1.721%	31.385%	29.828%	27.516%	5.921%	10.128%	0.585
MaxSharpe	11.272%	46.949%	43.954%	34.531%	16.636%	16.180%	1.028

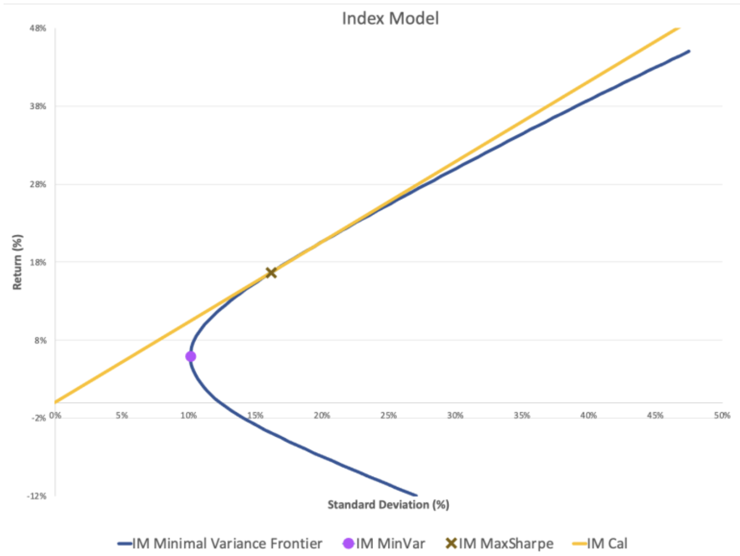

Fig. 4 Frontier, optimal portfolio and capital allocation line of IM under constraint 2

Constraint 3: This constraint does not set any restrictions. The aim is to observe the portfolio region and the efficient frontier that the chosen portfolio can form without setting any constraints. This is beneficial for subsequent comparative analysis follows and the related results are shown in Table 7 and 8, Figs 5 and 6.

Table 7. Weights of each asset of MM under constraint 3

MM	SPX	NVDA	CSCO	INTC	GS	USB	TD CN
MinVar	48.407%	-2.717%	-3.465%	4.031%	-6.297%	4.657%	1.807%
MaxSharpe	-194.148%	39.030%	6.456%	-17.320%	26.261%	20.723%	49.795%
	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	-12.337%	24.395%	24.894%	16.625%	6.209%	11.535%	0.538
MaxSharpe	9.499%	49.513%	83.221%	26.971%	25.595%	23.420%	1.093

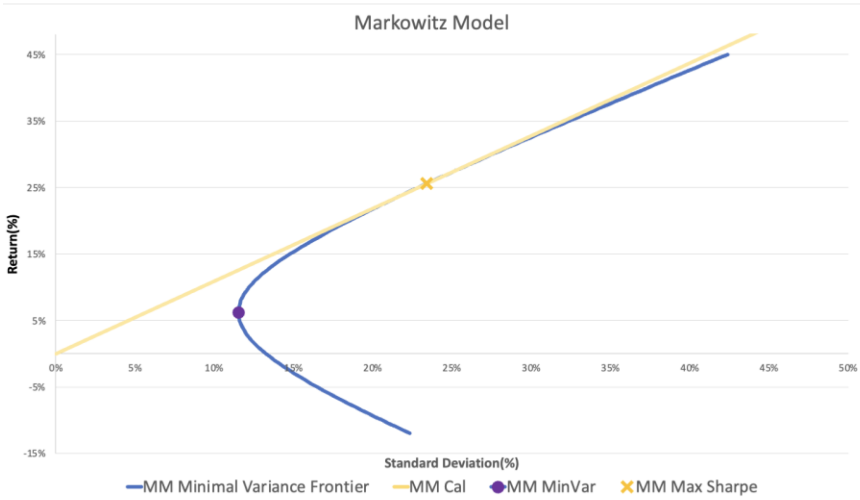

Fig. 5 Frontier, optimal portfolio and capital allocation line of MM under constraint 3

Table 8. Weights of each asset of IM under constraint 3

IM	SPX	NVDA	CSCO	INTC	GS	USB	TD CN
MinVar	32.907%	-4.186%	-4.283%	-0.340%	-9.746%	0.418%	-1.780%
MaxSharpe	-135.995%	20.156%	6.685%	-1.623%	9.021%	7.238%	41.657%
	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	-1.721%	31.385%	29.828%	27.516%	5.921%	10.128%	0.585
MaxSharpe	14.818%	51.244%	48.908%	37.892%	18.475%	17.891%	1.033

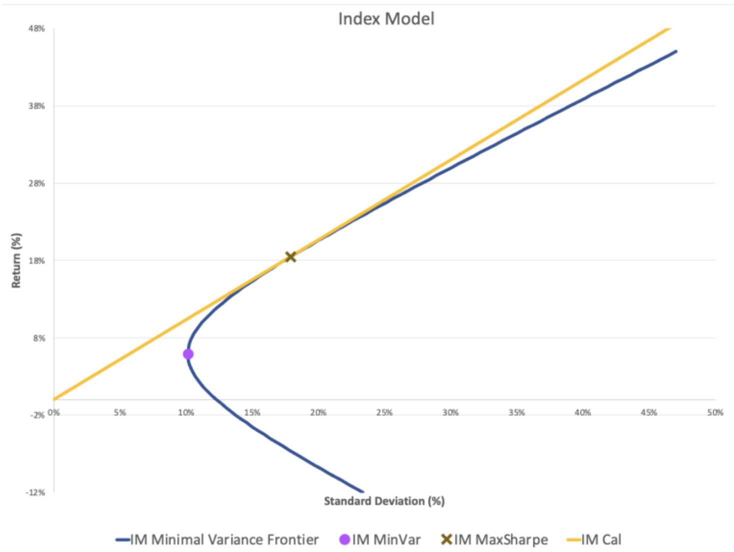

Fig. 6 Frontier, optimal portfolio and capital allocation line of IM under constraint 3

Constraint 4: Compared with ordinary investment, short selling has higher marginal cost and greater risk. In the financial market, ordinary investors still account for a large proportion, and they do not have enough knowledge to conduct short positions. In addition, the financial market system in some areas is not perfect enough. Therefore, in order to reduce the risk that investors need to bear, the Investment Company Act proposed by the United States in 1940 stipulates that open-end funds do not allow short positions. That is, this constraint requires that customers are not allowed to short, which is expressed as follows and the related results are shown in Table 9 and 10, Figs 7 and 8:

$$w_i \geq 0, for\ \forall i \tag{10}$$

Table 9. Weights of each asset of MM under constraint 4

MM	SPX	NVDA	CSCO	INTC	GS	USB	TD CN
MinVar	24.537%	0.000%	0.000%	2.277%	0.000%	2.832%	0.000%
MaxSharpe	0.000%	16.770%	0.000%	0.000%	0.000%	0.000%	18.594%
	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	0.000%	27.096%	24.545%	18.714%	7.711%	11.866%	0.650
MaxSharpe	0.000%	29.660%	26.287%	8.689%	13.729%	15.071%	0.911

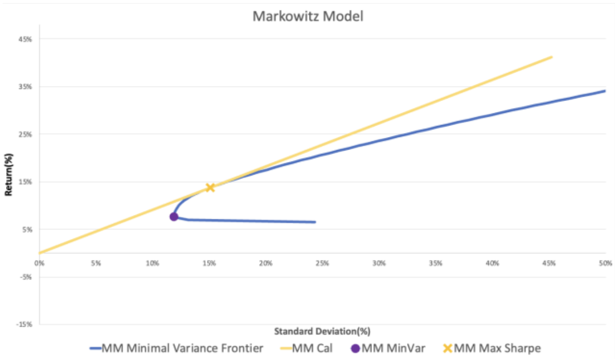

Fig. 7 Frontier, optimal portfolio and capital allocation line of MM under constraint 4

Table 10. Weights of each asset of IM under constraint 4

IM	SPX	NVDA	CSCO	INTC	GS	USB	TD CN
MinVar	2.902%	0.000%	0.000%	0.000%	0.000%	0.437%	0.000%
MaxSharpe	0.000%	12.384%	0.000%	0.000%	0.00%	0.000%	15.905%
	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	0.000%	34.192%	32.492%	29.978%	7.981%	10.572%	0.755
MaxSharpe	0.000%	30.668%	22.321%	18.722%	12.340%	13.721%	0.899

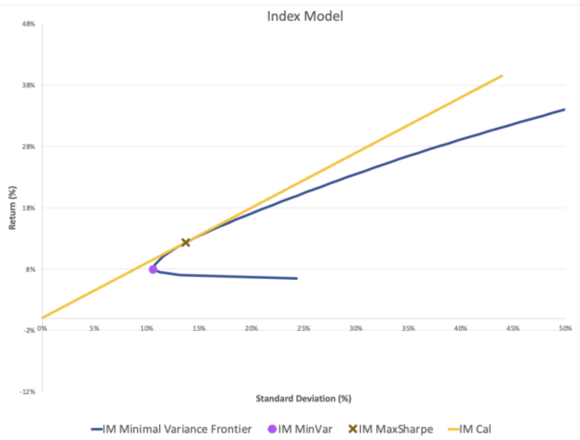

Fig. 8 Frontier, optimal portfolio and capital allocation line of IM under constraint 4

Constraint 5: The broad market index is used by most investors as a benchmark to measure the performance of stocks. Funds that invest in a broad index can increase the diversification of the investment portfolio and reduce investment risks, so they are welcomed by investors. The purpose of this constraint is to investigate what positive or negative effects there will be when the broad index is not included in the portfolio and the related results are shown in Table 11 and 12, Figs 9 and 10.

$$w_M = 0 \tag{11}$$

Table 11. Weights of each asset of MM under constraint 5

MM	SPX	NVDA	CSCO	INTC	GS	USB	TD CN
MinVar	0.000%	0.290%	0.507%	5.819%	0.100%	10.336%	6.171%
MaxSharpe	0.000%	21.782%	-7.535%	-18.600%	0.508%	0.355%	27.213%
	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	-9.893%	28.564%	35.556%	22.550%	8.200%	11.898%	0.689
MaxSharpe	-2.165%	31.970%	39.505%	6.969%	15.780%	16.506%	0.956

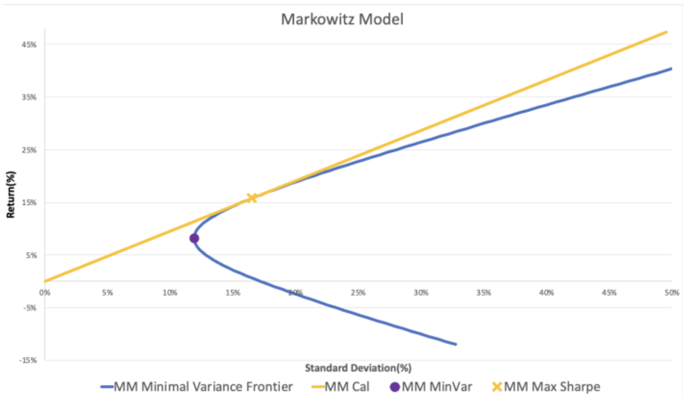

Fig. 9 Frontier, optimal portfolio and capital allocation line of MM under constraint 5

Table 12. Weights of each asset of IM under constraint 5

IM	SPX	NVDA	CSCO	INTC	GS	USB	TD CN
MinVar	0.000%	-3.298%	-1.148%	1.940%	-6.429%	3.688%	2.176%
MaxSharpe	0.000%	14.003%	-5.625%	-9.414%	-4.904%	-5.027%	22.402%
	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	1.500%	35.314%	34.916%	31.341%	7.016%	10.309%	0.681
MaxSharpe	1.506%	35.049%	28.763%	23.248%	13.080%	14.076%	0.929

Fig. 10 Frontier, optimal portfolio and capital allocation line of IM under constraint 5

By observing the above results, we can know that the investment portfolio without any investment constraints, i.e., constraint 3, outperforms the portfolios under other constraints. This may be because in order to meet the constraint requirements, some assets need to be reduced or even discarded or some assets need to be increased. In the case where short selling is not allowed, i.e., constraint 4, the maximum value of the Sharpe ratio is the smallest among the five constraints, and the standard deviation at this time is also the smallest. It shows that under the constraint that short selling is not allowed, both investment risk and return rate are reduced. Besides, the investment range of constraint 4 is greatly reduced, which is the smallest among the five constraints. This is a market protection measure for vulnerable buyers to help reduce risks. Observe that when the Sharpe ratio is the largest under all constraints, the proportion of S&P 500 is the smallest proportion in the investment portfolio, usually negative or 0. Comparing the proportion of S&P500 and the value of standard deviation in the case of minimal variance and maximal Sharpe ratio, not investing in the broad market index can negatively affect the portfolio. It can be concluded that the broad market index can indeed reduce the risk of investment portfolio. In addition, according to table 3 to 12, it can be known that when calculating the maximal Sharpe ratio the Markowitz model performs better, and when calculating the minimal variance, the index model performs better.

5. Conclusion

This research combines financial theory and puts it into practice and conducts portfolio analysis on representative companies in consumer defensive, technology, financial service and healthcare industries. This is because most of the studies are aimed at the general situation of the market or specific industries, so this paper selects several industries with less research for portfolio analysis. By setting different restrictions to simulate the financial market environment and the needs of different investors, it aims to help investors make better investment decisions. Eleven assets are analyzed under five additional constraints by using two models. This paper finds that returns decrease, and risks increase after constraints are added. Therefore, investors should understand the market rules before investing, and then adjust their investment strategies according to their own investment preferences.

However, there are also some deficiencies in this paper. In the actual financial market, the quality of investors is mixed, and the requirements of each person are also very different. Compared with this, the Markowitz model only considers the most ideal situation. This article only discusses stocks as asset, but there are other investable assets that investors can choose from. Besides, what impact some special constraints will have on investment remains to be studied.

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