An Empirical Study on the Markowitz portfolio

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Abstract. Markowitz's portfolio model laid the foundation of the modern portfolio theory (MPT). As an intuitive and effective quantitative method, the Markowitz model has long been considered one of the most successful approaches in financial modeling. This paper tries to illustrate the process of Markowitz modeling and demonstrate the usefulness of the Markowitz theory empirically. Three high-tech companies, Apple Inc. (AAPL), Alphabet Inc. (GOOG), and Microsoft Corporation (MSFT) investigated to build the Markowitz model. We first retrieve the dataset from the website of yahoo finance and provide the descriptive statistics for the three companies, respectively. Then, the global minimum variance (GMV) portfolio is constructed to identify the boundary point of the efficient frontier. The optimized portfolio model at the given risk level is constructed according to the Markowitz theory. Also, the random portfolio weights under the budget constraint are generated for comparison. To evaluate the performance, we provide the Sharpe ratio for the constructed Markowitz portfolio.

Keywords: Markowitz Model; Portfolio Theory; Efficient Frontier.

1. Introduction

Markowitz's theory has gained tremendous theoretical success since it was proposed by Harry Markowitz [1]. Therefore, the Markowitz model is viewed as the inception of the modern portfolio theory, and lots of variants have been developed based on the classical portfolio model.

As an intuitive and effective portfolio model, Markowitz's portfolio uses the mean value representing the return of the involved risky assets, and the variance representing the risk of the selected risky assets. Hence, the Markowitz portfolio model is also called the mean-variance portfolio model (MV) by some scholars. Essentially, the seminal work of Markowitz's portfolio theory introduces an idea of a trade-off between risk (variance) and the expected return of a portfolio. The Pareto-optimal subset of portfolios on the efficient frontier, that is, the expected returns of these portfolios would not increase unless their risk levels increase, or vice versa. To get the efficient frontier, a series of quadratic programming should be solved.

However, Markowitz's portfolio requires strong assumptions on the probability distributions of the risky assets returns [2]. Additionally, solving the corresponding quadratic programming is time-consuming if there exists a large number of candidate risky assets. Although the Markowitz model remains the general financial framework, some shortcomings of the static portfolio model have aroused the interest of practitioners and researchers. Even now some heated discussions about the pragmatic value of the Markowitz portfolio still continue in lots of academic journals.

The static Markowitz portfolio tends to derive unstable asset allocations which are quite sensitive to small changes from the underlying estimated parameters [3]. Some artificial intelligence techniques such as machine learning applications provide some insights into constructing automatic portfolio allocations, that is, robo-advisors [4]. Optimization tools are also noteworthy for polishing the traditional MV portfolio model. Two fundamental methods could be distinguished from the literature [5]. The first method is to adjust the objective function by adding the regularizers [6], by which the portfolio weights could be controlled. The second method is to drop the primal MV objective function and turn to other objective functions. For example, the risk parity portfolio aims to build a portfolio with equal risk contribution [7]. The most diversified portfolio with the objective to diversify risk as much as possible [8].

This paper aims to illustrate the modeling process of the static Markowitz portfolio, where three high-tech companies are involved. This study first constructs the global minimum variance portfolio, then the optimized Markowitz portfolio and the efficient frontier are implemented, in which some optimized problems are solved using MATLAB and Gurobi. According to the empirical results, the risky assets with high return and low volatility would be favored by the Markowitz portfolio, thus the distribution of the optimized portfolio weight is imbalanced.

The structure of the rest is as follows. Section 2 presents and describes the used dataset, Section 3 constructs the Markowitz portfolio, Section 4 presents the analysis and discussions on the model, and Section 5 concludes the paper.

2. Data

Three high-tech companies listed on the S&P500 index, AAPL (Apple Inc.), GOOG (Alphabet Inc.), and MSFT (Microsoft Corporation) are considered in this paper. We retrieve the data ranging from Dec. 26, 2021, to Dec. 26, 2022, on the website of yahoo finance. The three companies belong to the high-tech industry, where AAPL focuses on electronics manufacturing, GOOG focus on the Internet, and MSFT focuses on software.

AAPL designs manufacture and markets phones (iPhone), and personal computers (MacBook) worldwide. The quarterly financial information about AAPL is shown in Table 1. The Beta (5Y monthly) is 1.22, the revenue of AAPL is 394.33B with 24.32 revenue per share, the quarterly revenue growth is 8.10%, and the gross profit is 170.78B with 130.54B EBITDA. Fig. 1 shows the candle chart of AAPL.

Table 1. quarterly financial information of AAPL.

	9/30/2022	6/30/2022	3/31/2022
Market Cap.	2.20T	2.20T	2.83T
Enterprise Value	2.27T	2.27T	2.89T
Trailing P/E	22.84	22.23	28.96
Price/Sales	5.87	5.87	7.72
Enterprise Value/Revenue	25.24	27.35	29.70
Enterprise Value/EBITDA	80.24	85.33	86.06

Table 2. quarterly financial information of GOOG.

	9/30/2022	6/30/2022	3/31/2022
Market Cap.	1.24T	1.43T	1.84T
Enterprise Value	1.15T	1.32T	1.72T
Trailing P/E	17.91	19.79	24.89
Price/Sales	4.63	5.45	7.35
Enterprise Value/Revenue	16.61	18.97	25.36
Enterprise Value/EBITDA	56.30	57.49	75.65

Table 3. quarterly financial information of MSFT.

	9/30/2022	6/30/2022	3/31/2022
Market Cap.	1.74T	1.92T	2.31T
Enterprise Value	1.69T	1.87T	2.25T
Trailing P/E	24.13	19.79	24.89
Price/Sales	8.86	5.45	7.35
Enterprise Value/Revenue	33.78	36.12	45.50
Enterprise Value/EBITDA	68.11	75.06	91.79

GOOG operates various platforms such as Google Services, Google Cloud, and other Bets segments. The quarterly financial information about GOOG is shown in Table 2. The Beta (5Y monthly) is 1.06, the revenue of GOOG is 282.11B with 21.45 revenue per shape, the quarterly revenue growth is 6.10%, and the gross profit is 146.7B with 93.73B EBITDA. Fig. 2 shows the candle chart of GOOG.

MSFT develops and licenses software, services, devices, and solutions worldwide. The quarterly financial information about MSFT is shown in Table 3. The Beta (5Y monthly) is 0.93, illustrating that MSFT is more conservative than the market. The revenue of MSFT is 203.07B with 27.14 revenue per share, the quarterly revenue growth is 10.60%, and the gross profit is 135.62B with 98.84B EBITDA. Fig. 3 shows the candle chart of MSFT.

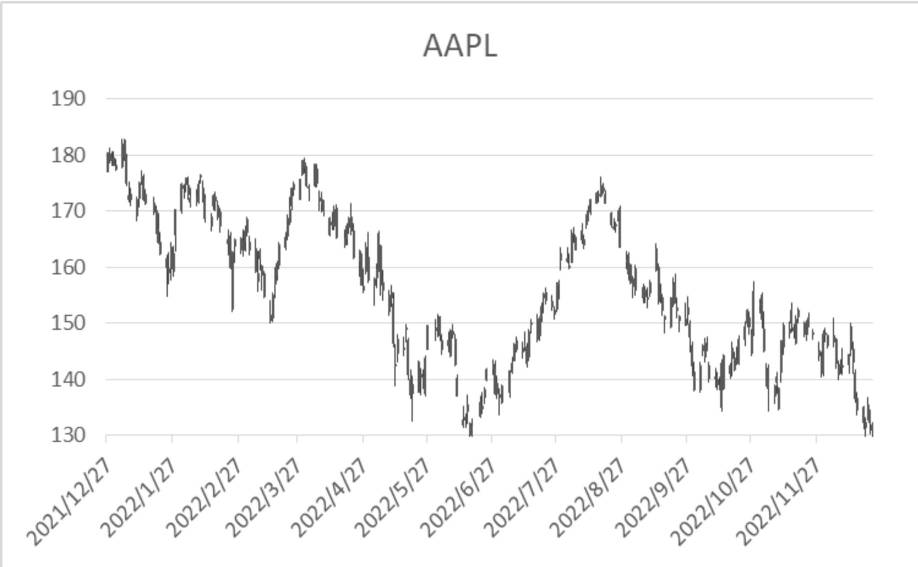

Figure 1. candle chart of AAPL (from yahoo finance).

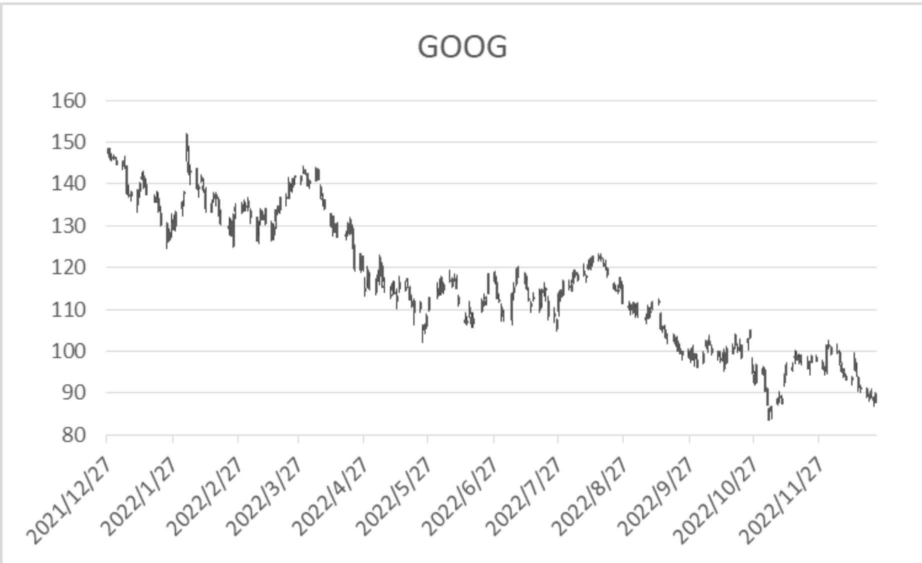

Figure 2. candle chart of GOOG (from yahoo finance).

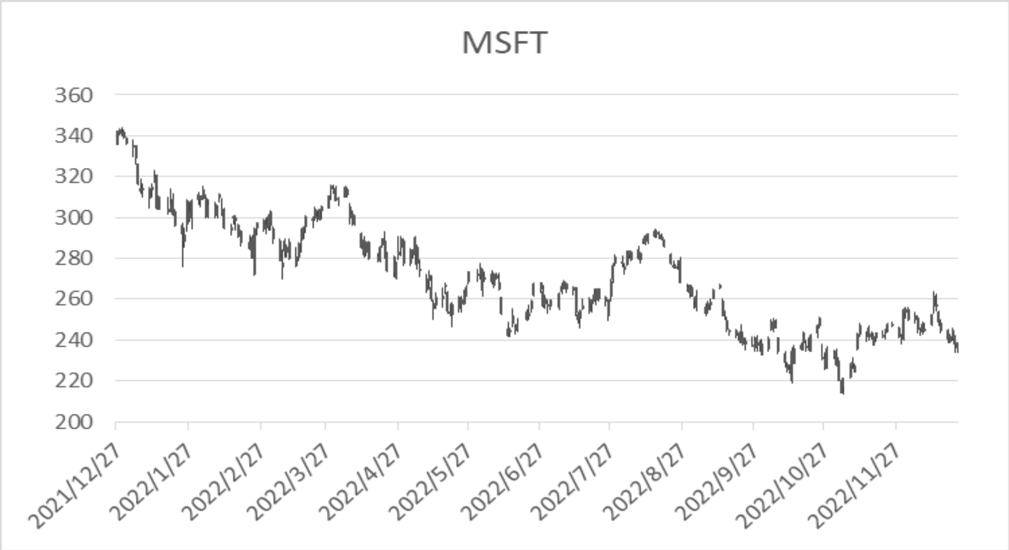

Figure 3. candle chart of MSFT (from yahoo finance)

The descriptive statistics for AAPL and GOOG are shown in Table 4. Fig. 4 presents the curves of the close price for AAPL, GOOG, and MSFT, respectively. Fig. 5 presents the curves of the daily return for AAPL, GOOG, and MSFT, respectively.

From the table of descriptive statistics, AAPL has a higher mean return than GOOG and MSFT, and also with the highest max return and return. Obviously, the economic environment for the high-tech industry is not quite ideal since all of the involved companies show negative mean returns in 2022. In terms of risk, MSFT has the lowest standard deviation (Stdev.), demonstrating the attribute of robustness for MSFT.

Table 4. descriptive statistics for the returns of AAPL and GOOG.

	AAPL	GOOG	MSFT
Mean return	-0.12%	-0.2%	-0.14%
Stdev.	2.23%	2.43%	2.22%
Min return	-6.05%	-10.13%	-8.03%
Max return	8.52%	7.46%	7.91%

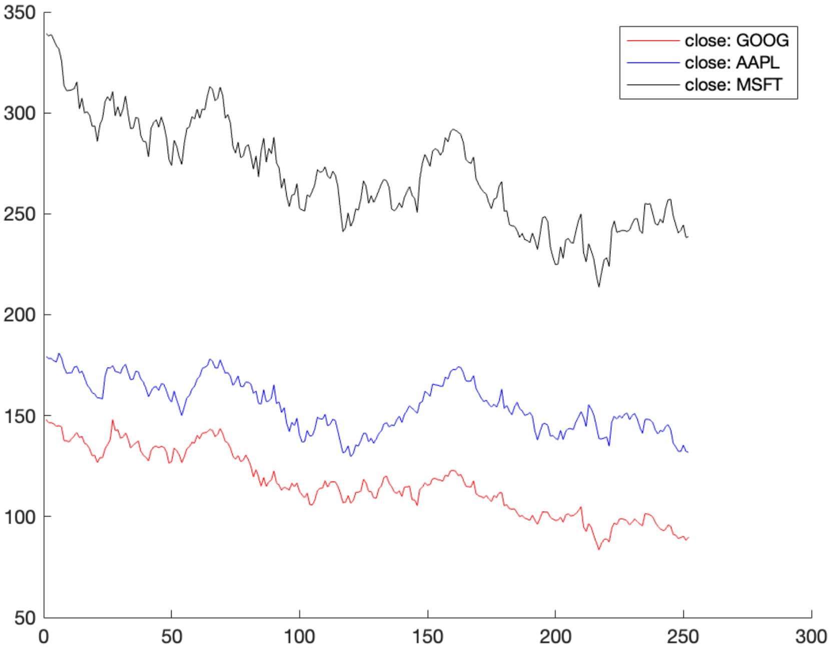

Figure 4. close prices for AAPL, GOOG and MSFT.

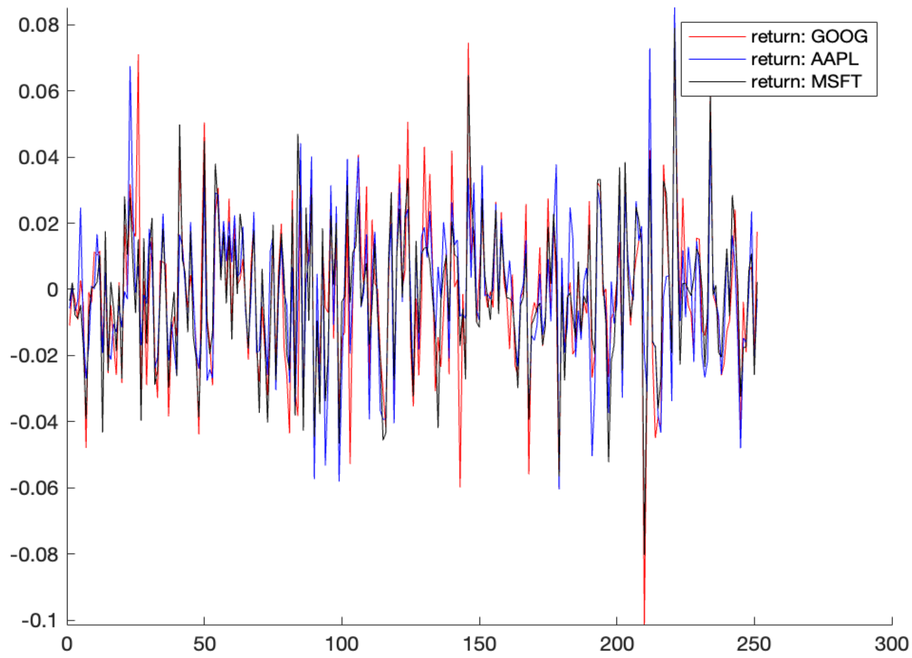

Figure 5. daily return for AAPL, GOOG and MSFT

3. Portfolio Construction

Portfolio theory aims to build a numeric model for some risky assets. By solving the corresponding portfolio model, the optimized weight vector could be obtained. Two problems should be investigated in constructing the portfolio model. The first one is which numeric model should be selected. Lots of quantitative methods such as CPAM [9], Markowitz model [10], Fama-French [11-13], APT model[14] are effective in building portfolios. The second one is how to solve the constructed portfolio model. If the problem is convex, then some solvers such as Gurobi [15], CPLEX [16], Mosek, and SCIP are useful in dealing with the model. However, when some complex, dynamic constraints are incorporated into the framework of the portfolio, solving the model becomes an NP-hard problem. Therefore, some heuristic algorithms are beneficial to solving these problems.

This section constructs the portfolio model according to the famous Markowitz theory in the following, in which the expectation of return represents the earnings of a stock, while the variance or standard deviation represents the risk level of a stock. The classical Markowitz portfolio is an intuitive parametric model, which laid the foundation for modern portfolio theory [10].

3.1 Markowitz Model

Assumes that a portfolio with N risky assets, in which the return vector of asset i is $R_i = \{r_1, r_2, \dots, r_T\}$. Define μ_i as the mean return of R_i, σ_i^2 as the variance of R_i, and σ_{ij} as the covariance between R_i and R_j. Suppose the weight invested on the ith risky asset is x_i, the Markowitz portfolio with the non-shorting constraint is as follows:

$$\mu = E(R) = \sum_{i=1}^{N} \mu_i x_i \quad (1)$$

$$\sigma^2 = V(R) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} x_i x_j \quad (2)$$

$$\sum_{i=1}^{N} x_i = 1 \quad (3)$$

$$x_i \geq 0, i = 1, 2, \dots, N \quad (4)$$

For different combinations of portfolio weight, the attainable set of (σ^2, μ) could be obtained, in which the (σ^2, μ) with the minimum portfolio risk level for a given portfolio return target, or the (σ^2, μ) with the maximum portfolio return level for a given portfolio risk target are on the efficient frontier.

In this case, $N = 3$, and the variance-covariance matrix of the risky assets returns is:

$$\begin{bmatrix} 0.0496\% & 0.0426\% & 0.0406\% \\ 0.0426\% & 0.0592\% & 0.0455\% \\ 0.0406\% & 0.0455\% & 0.0492\% \end{bmatrix} \tag{5}$$

And the correlation coefficient matrix is:

$$\begin{bmatrix} 1.0000 & 0.7865 & 0.8217 \\ 0.7865 & 1.0000 & 0.8443 \\ 0.8721 & 0.8443 & 1.0000 \end{bmatrix} \tag{6}$$

Assumes that the investors hold the equal-weighted portfolio (EWP), that is, $X = (1/3, 1/3, 1/3)'$, the return of EWP is $X'\Sigma X = 0.0462\%$. To obtain the portfolio with minimal variance (GMV), the following problem should be solved:

$$min\, X'\Sigma X \tag{7}$$

$$s.t.\, \mu' X \geq \mu_{ew} \tag{8}$$

$$\sum_{i=1}^{N} x\_i = 1 \tag{9}$$

Using Yalmip [17] in MATLAB [18], the objective value of GMV is 0.045%, and the corresponding portfolio weight vector is $(0.4686, 0.0562, 0.4752)'$. Note that we slack the non-shorting constraint to get the global minimum variance portfolio, but the solution also satisfies the non-shorting constraint.

3.2 Portfolio optimization

Define the variance of GMV is σ_G^2, the optimized portfolio could be obtained by solving the following optimization problem:

$$max\, \mu' X \tag{10}$$

$$s.t.\, X'\Sigma X \geq \sigma_G^2 \tag{11}$$

$$\sum_{i=1}^{N} x_i = 1 \tag{12}$$

$$x_i \geq 0 \tag{13}$$

Using the solver as before, the optimized portfolio weight vector is $(0.5199,\ 0.0095,\ 0.4707)'$, and the return of the optimized portfolio is -0.0013, the corresponding variance of the optimized portfolio is 0.045%. From Table 4, all of the stocks show negative mean return, hence the optimized portfolio return also is negative, but the variance could be reduced using the optimized portfolio.

To illustrate the effectiveness of the optimized portfolio, 10000 random sample portfolio weight vectors are generated to show the efficient frontier.

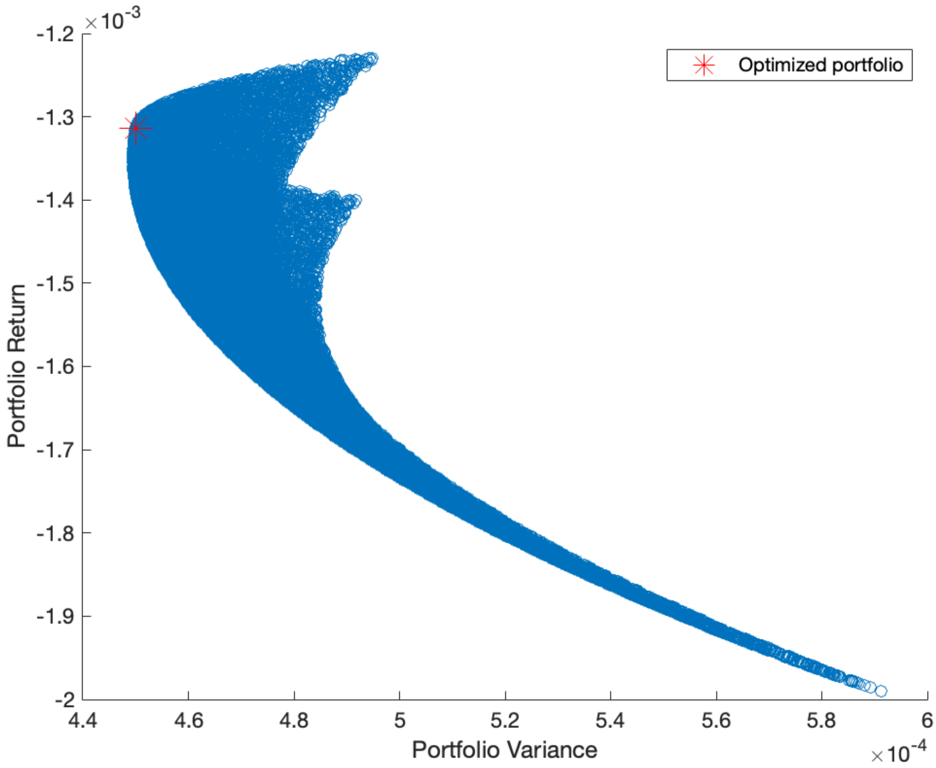

Figure 6. efficient frontier.

Fig. 6 shows the efficient frontier of the Markowitz model, where the portfolio weights on the upper part are superior to the ones on the lower part because for given portfolio variance, the upper part portfolios show greater portfolio return. Therefore, the upper boundary constitutes the efficient frontier, and the optimized portfolio and the GMV portfolio are on the efficient frontier. Table 5 presents detailed information on the optimized portfolio.

Table 5. Information on the optimized portfolio.

	Value
AAPL	51.99%
GOOG	0.95%
MSFT	47.07%
Sharpe ratio (annual)	-1.47

Sharpe ratio [19] is one of the widely used indicators to evaluate the performance of a portfolio model, with the formula as follows:

$$SR = \frac{R_p - R_f}{\sigma_p} \tag{14}$$

Where the risk-free is 3% per annual according to the T-Bill, the greater the Sharpe ratio is, the more satisfied performance a portfolio would show. Due to the Covid-19 and macroeconomic impacts, AAPL, GOOG, and MSFT show negative returns over the whole of 2022, Sharp ratio cannot gauge the portfolio performance exactly.

4. Analysis & Discussions

From Table 4, GOOG shows the lowest return level and the highest risk level, which is unwelcome by rational investors, and also by the Markowitz model. The results in Table 5 demonstrate the conclusion, in which the portfolio weight for GOOG is only 0.95%, the optimized portfolio model distributes the highest weight to AAPL, 51.99%, and the second highest weight to MSFT, 47.07%.

The greatest advantage of the Markowitz model is that the parametric model provides an intuitive and qualitative method for modeling. Without some complex constraints, the constructed model could be solved efficiently.

Nonetheless, some drawbacks of the Markowitz model are also presented by lots of researchers. Parameter sensitivity is one of the biggest issues discussed by scholars, that is, when some little disturbance in the estimated parameters, the optimized weights could change significantly. The parameters are estimated from the historical samples, which is error-prone to future trends, and the process of optimization could exacerbate the estimation error, which would result in poor out-of-sample performance.

Black-Litterman portfolio model [20] was demonstrated to be an effective improvement based on the Markowitz model. In the Black-Litterman model, expert views could be combined with the parameters from sample estimation in the Bayes framework, and some machine learning tools could be used for generating accurate estimations. Numeric experiments have verified that the Black-Litterman portfolio model could reduce the parameter sensitivity to some extent.

5. Conclusion

This paper constructs the classical Markowitz portfolio with detailed steps. AAPL, GOOG, and MSFT are the three high-tech companies considered in the process of modeling. Due to the influences of Covid-19 and worldwide macroeconomics, all the candidate stocks show negative expected returns, which result in the negative Sharpe ratio (SR) of the constructed Markowitz portfolio. However, the negative Sharpe ratio could not reflect the portfolio performance accurately and effectively.

The efficient frontier could be obtained using the randomly generated portfolio weights. The minimal variance portfolio could be used to locate the leftmost point of the portfolio efficient frontier, and the optimized Markowitz portfolio is ought to show on the efficient frontier. Rational investors should prefer the portfolios on the efficient frontier because these portfolios have higher expected returns than those below the efficient frontier for the given risk levels.

The non-shorting constraint is usually considered during portfolio modeling. Without the budget constraint, the optimized portfolio weight could be negative, which means shorting these risky assets. To be consistent with the customized rules of the real stock market, researchers would add the non-shorting constraint when modeling the stock market, but this constraint could be slacked when modeling the future market, in which bi-directional trading is allowed. From the results of the Markowitz portfolio, the portfolio weight is not diversified enough since AAPL and MSFT show significantly higher portfolio weights than GOOG. Nonetheless, some conservative investors prefer balanced portfolios which would diversify the non-systematic risk to a large extent. For these investors, the Markowitz portfolio is not the most suitable model, but the risk parity model is more appealing instead.

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