The Unique Characterization of the Shapley Value for Bi-cooperative Games

Meirong Wu [1], Wenbo Wang [2]

[1] The Department of Basic Courses, Army Logistics University, 4300030, Chongqing, China.

[2] College of Science, Wuhan University of Science and Technology, Wuhan Hubei, 430064, China

Abstract. The aim of the present paper is to study the unique characterization for bicooperative games. Shapley value is the expected marginal contribution of the alliance. We will introduce some properties for bicooperative games. Our first characterization is based on the classical axioms determining the Shapley value with the symmetry axiom replaced excluded null axiom. In our second axiomatization we use structural axiom and a zero excluded axiom instead of effective axiom in classical cooperative games. Finally, We provide here linearity, anonymity, dummy and efficiency and structural axioms to study a one-point solution concept for Bi-cooperative games.

Keywords: Shapley value Bi-cooperative Games; Characterization.

1. Introduction

In a cooperative game (N,v), where $N=\{1,2,\cdots,n\}$ is a finite set of players and the symbol v represents a real value function assigning each subset S of $N, v:2^N \to R, R$ denotes the real number set. For each $S \in 2^N$, the worth $v(S)$ can be seen as the maximal revenue or minimal cost that the players who form the coalition S can achieve themselves against the best offensive threat by the complementary coalition $N \setminus S$ [6]. We consider pairs (S,T) with $S,T \subseteq N$ and $S \cap T = \varphi$. Thus, (S,T) yields a partition of the set N of all players in three groups. Players in S are defenders of modifying the actual situation and they want to accept a proposal; players in T do not agree with modifying the situation and they will take action against any change. Finally, the members of $N \setminus (S \cup T)$ are not convinced of the profits derived from the proposal, but they do not think of objecting and stop the action managed by the elements of coalition S.

Similarly to the cooperative case, we consider the set of all ordered pairs of disjoint coalitions $3^N = \{(S,T): S,T \subseteq N, S \cap T = \varphi\}$ and define a function $B:3^N \to R$, For each $(S,T) \in 3^N$ the worth $b(S,T)$ represents the maximal revenue (whenever $b(S,T) > 0$) or minimal loss (whenever $b(S,T) < 0$) that is obtained when S is the set of players for a change in the activity, the players of T are against the change and the players of $N \setminus (S \cup T)$ are not taking part because they are indifferent. This leads us in a natural way into the concept of bicooperative game introduced by Bilbao [1].

Definition 1.1. A bicooperative game is a pair (N,b) with N a finite set and b a function $B:3^N \to R$, with $b(\varnothing,\varnothing)=0$.

In order to convenient writing, we use $S \cup i$ and $S \setminus i$ instead of $S \cup \{i\}$ and $S \setminus \{i\}$ in this paper. This paper will search for a n-dimensional vector $b(x_1,\cdots x_n)$, x_i assigned to each player revenue in cooperative games. Shapley value is a common solution. Shapley value usually assumes that each player in the game is the same size as possible to join the coalition. Shapley value is the expected marginal contribution of the alliance. In the next section, this paper introduces some properties for bicooperative games, The aim of the third section is to obtain axiomatizations of the Shapley value for bicooperative games. Four of them are extensions of the classical axioms for the Shapley value: linearity, anonymity, dummy and efficiency [3], but they don't characterize the Shapley value for bicooperative games. Meanwhile, we introduced a structural axiom and a zero excluded axiom instead of effective axiom in classical cooperative games. We finish the uniqueness of the bicooperative game description.

2. The properties for bicooperative games

Let $N=\{1,2,\cdots,n\}, 3^N=\{(A,B):A,B\subseteq N, A\cap B=\varphi\}$. Grabisch and Labreuche proposed a relation in 3^N given by $(A,B)\subseteq(C,D)\Leftrightarrow A\subseteq C, B\supseteq D$.

The set $(3^N,\subseteq)$ is a partially ordered set with the following properties:

1. (φ,N) is the first element: $(\varphi,N)\subseteq(A,B)$ for all $(A,B)\in 3^N$.

2. (N,φ) is the last element: $(A,B)\in(N,\varphi)$ for all $(A,B)\in 3^N$.

3. For every pair $\{(A,B),(C,D)\}$ in 3^N,

$$(A,B)\vee(C,D)=(A\cup C, B\cap D),$$
$$(A,B)\wedge(C,D)=(A\cap C, B\cup D).$$

Proposition 2.1. The number of maximal chains of 3^N, the following is true:

(a) The number of maximal chains of 3^N is $(2n)!/2^n$, where $n=|N|$.

(b) For all $(A,B)\in 3^N$, the number of maximal chains of the sublattice $[(\varphi,N),(A,B)]$ is $(n+a-b)!/2^a$,where $a=|A|, b=|B|$.

(c) Let $(A,B),(C,D)\in 3^N$ with $(A,B)\subset(C,D)$, The number of maximal chains of the sublattice $[(A,B),(C,D)]$ is equal to the number of maximal chains of the sublattice $[(D,C),(B,A)]$.

The proof is seen [1] and [4].

3. The Shapley value for bicooperative games

Definition 3.1. The Shapley value for the bicooperative game $b\in BG^N$ is a vector $\Phi\in R^N$ defined for each $i\in N$, by

$$\Phi_i^{sh}(b)=\sum_{(S,T)\in 3^{N\setminus i}}\left[\overline{p}_{s,t}(b(S\cup i,T)-b(S,T))+\underline{P}_{s,t}(b(S,T)-b(S,T\cup i))\right]$$

where, for all $(S,T)\in 3^{N\setminus i}$,

$$\overline{p}_{s,t}=\frac{(n+s-t)!(n+t-s-1)!}{(2n)!}2^{n-s-t}, \underline{p}_{s,t}=\overline{p}_{s,t}=\frac{(n+t-s)!(n+s-t-1)!}{(2n)!}2^{n-s-t}$$

With the aim to characterize the Shapley value for bicooperative games, we consider a set of reasonable axioms and we prove that the Shapley value is the unique value on BG^N which satisfies these axioms.

Linearity axiom: For all $\alpha,\beta\in R$ and $b,w\in BG^N$,

$$\Phi_i(\alpha b+\beta w)=\alpha\Phi_i(b)+\beta\Phi_i(w), \text{ for all } i\in N$$

A player i is a zero player when he have no contribution to coalitions. For every $b\in BG^N$, $(S,T)\in 3^{N\setminus i}$, the following is true.

$$b(S\cup i,T)-b(S,T)=0,$$
$$b(S,T)-b(S,T\cup i)=0,$$

Thus, if $i\in N$ is a zero player, $\forall (S,T)\in 3^{N\setminus i}$,

$$b(S\cup i,T)-b(S,T\cup i)=0,$$

Zero axiom: For $b\in BG^N$, $i\in N$ is a zero player,

$$\Phi_i(b)=0$$

Anonymity axiom: For all $b\in BG^N$,if $i\in N$ and for any permutation π over N, it holds that

$$\Phi_{\pi i}(\pi b)=\Phi_i(b) \text{ for all } i\in N, \text{ where } \pi b(\pi S,\pi T)=b(S,T) \text{ and } \pi S=\{\pi i: i\in S\}.$$

The value of player isn't depending on the labels of the players for anonymity axiom.

Excluded null axiom: For all $b\in BG^N$, if $i\in N$ is the zero player for b,then $\forall j\in N\setminus i$,

$$\Phi_j(b) = \Phi_j(b^{-i})$$

The zero player leave the games, the others have similar value in associated constricted games [2].

Lemma 3.2. The Shapley value on $b \in BG^N$ satisfies the Linearity axiom, Zero axiom, Anonymity axiom and Excluded null axiom.

For lemma 3.2, the Shapley value on $b \in BG^N$ satisfies the Linearity axiom, Zero axiom Anonymity axiom, but the value is not unique. For example,

$$\Phi_i^{sh}(b) = \sum_{S \in N \setminus i} \frac{s!(n-s-1)!}{n!} \left[b(S \cup i, N \setminus (S \cup i)) - b(S, N \setminus S) \right]$$

satisfies that four axioms too. Notice that it is similar to the value for cooperative games where $v: 2^N \to R$ defines: Let $A = \varnothing$ and $v(\varnothing) = 0$, then $v(A) = b(A, N \setminus A)$ .but it don't satisfy the bicooperative games. Be similar to Weber [3], we take into account structural axiom.

Theorem 3.3. Let Φ_i is the value of player i, Φ_i satisfies the Linearity axiom , Zero axiom and Anonymity axiom, for all $b \in BG^N$,

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} \left[\overline{p}_{s,t}(b(S \cup i, T) - b(S,T)) + \underline{p}_{s,t}(b(S,T) - b(S, T \cup i)) \right]$$

If Φ_i satisfies the anonymity axiom,then $\overline{p}^i_{(S,T)} = \overline{p}_{s,t}, \underline{p}^i_{(S,T)} = \underline{p}_{s,t}$,for all $(S,T) \in 3^{N \setminus i}$.

Theorem 3.4. Let Φ_i is the value of player i, Φ_i satisfies the Linearity axiom , Zero axiom, Anonymity axiom and Excluded null axiom , for all $b \in BG^N$,

$$\Phi_i(b) = \sum_{(S,T) \in 3^{N \setminus i}} \left[\overline{p}_{s,t}(b(S \cup i, T) - b(S,T)) + \underline{p}_{s,t}(b(S,T) - b(S, T \cup i)) \right]$$

and

$$\overline{p}_{n-1,0} = \frac{1}{n}, \underline{p}_{0,n-1} = \frac{1}{n}$$

For all $0 \le s, t \le n-1$ and $0 < s + t \le n-1$,

$$(n-s-t)\overline{p}_{s,t} + t\underline{p}_{s,t-1} = (n-s-t)\underline{p}_{s,t} + s\overline{p}_{s-1,t}.$$

It is well known that the shapley value for bicooperative games satisfied the above four axioms is not characterized. From this point of view, our value must satisfy the Structural axiom. First, note that the signed coalitions $(S \setminus j, T)$ and $(S, T \cup i)$ (where $j \in S, i \notin S \cup T$) have the same rank $\rho[(S \setminus j, T)] = \rho[(S, T \cup i) = n + s - t - 1$. By Proposition 2.1,

$$c([(\varnothing, N), (S \setminus j, T)]) = \frac{(n+s-1-t)!}{2^{s-1}}$$

$$c([(\varnothing, N), (S \setminus j, T)]) = \frac{(n+s-1-t)!}{2^{s}}$$

Hence, the number of maximal chains in $[(\varnothing, N), (S \setminus j, T)]$ is not equal to $[(\varnothing, N), (S, T \cup i)]$ .Therefore the probability of two formation of (S,T) are distinct. (One is from $(\varnothing, N)$ to $(S \setminus j, T)$ to (S,T) ,Another is from $(\varnothing, N)$ to $(S, T \cup i)$ to (S,T).

In analogous form, if we consider $(S, T \setminus k)$ and $(S \cup i, T)$ (with $k \in T$)which has the same rank, the number of maximal chains in $[(S, T \setminus k), (N, \varnothing)]$ is not equal to number of maximal chains in $[(S \cup i, T), (N, \varnothing)]$ .Therefore the probability of formation of $(N, \varnothing)$ beginning from $(S, T \setminus k)$ to $(N, \varnothing)$ must be distinct to beginning from $(S \cup i, T)$ to $(N, \varnothing)$ .Taking into account these considerations, the values that one player must obtain in the identity games must be proportional to the number of maximal chains in the corresponding sublattices. It must be also considered that one value verifying the above four axioms assigns a non-negative real number to one player i in the identity game if this player belongs S and a non-positive real number if the player i belongs T .From this point of view, our value must satisfy the following axiom.

Structural axiom For every $(S,T) \in 3^{N \setminus i}, j \in S, k \in T$,

$$\frac{c([(\varphi, N), (S \setminus j, T)])}{c([(\varphi, N), (S, T \cup i)])} = -\frac{\Phi_j(\delta_{(S,T)})}{\Phi_i(\delta_{(S,T \cup i)})}, \frac{c([(S, T \setminus k), (N, \varphi)])}{c([(S \cup i), (N, \varphi)])} = -\frac{\Phi_k(\delta_{(S,T)})}{\Phi_i(\delta_{(S \cup i, T)})}$$

Theorem 3.5. Let Φ be a value on BG^N, The value Φ is the Shapley value if and only if Φ satisfies the linearity axiom, zero axiom ,excluded null axiom ,anonymity axiom and structural axiom.

Concluding remarks We were motivated by the introduction in which every player can choose with three different choices. We model these situations and one aim is how to define solution concepts. The shapley value in bi-cooperative games assumes every player has equal opportunities to join the alliance and all coalitions with the same size are equally. the players in $N \setminus S$ have no influence on the value of S if the coalition S is formed. For instance, a resident in a residential area in the proposed construction of a playground for children, some residents may support this proposal, some residents may oppose, some residents may be indifferent. These situations may be interpreted in the following manner. So we characterize initially the Shapley value of bicooperative games. For introducing a structural axiom and a zero excluded axiom instead of effective axiom in classical cooperative games.

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