Optimal Portfolio Construction Based on Markowitz Model

Qiyuan Feng*

College of Letter and Science, University of California, Santa Barbara, California, United States

*Corresponding author: qiyuanfeng@umail.ucsb.edu

Abstract. The appearance of Markowitz Model significantly improves the way investors optimize their financial portfolio, allowing them to reduce the collective risks of different assets and further maximize the profitability of their portfolio. This article aims to discuss and analyze the feasibility of Markowitz Model in practical cases as the definition of this mean variance model leans to the theoretical aspect of the finance like efficient market and rational investors. The data that applied to this study will be ten stocks from different types of companies and a market-capitalization index S&P 500. By utilizing the expected return and standard deviation formula, the efficient frontiers, minimum variance frontiers and max Sharpe portfolio are calculated respectively. Then simulating the practical limitations through applying five distinct constraints on the original results to take a glance at the performance of Markowitz Model in real cases. At last, discussing the restrictions of this theoretical model based on the data from this study and evidence that claimed by predecessor is necessary for demonstrating the preciseness. By comparing and contrasting the data under ideal situation and real cases, it reaches to the conclusion that the investors do gain higher returns and more optimized portfolio by means of the Markowitz Model.

Keywords: Markowitz Model; Empirical Study; Optimized Portfolio; Minimum Variance Frontiers, Max Sharpe Portfolio.

1. Introduction

Investments and selections of portfolio are always the central topics of how to maximize the profitability of people's assets. Since the emergence of modern financial portfolio like stocks, bonds, and commodities, there are countless of experts and analysts dedicated to the minimize the risks of investments and help investors wisely allocate the resources for their portfolio. In the early 1900s, where the securities were still at the growth stage and exhibited the tendencies of thriving, the experts in the finance industry tended to evaluate them separately and placed them inside the portfolio when each of the individual security did not expose excessive risk. It was Harry Markowitz in 1952, who won the 1990 Noble Prize in Economics, constructed a formal model of portfolio selection accompanied with principles of efficient diversification in his published article "Portfolio Selection", which overthrow the conventional perspectives on portfolio selection and later become a key concept in the foundation of advanced investment. Harry Markowitz claimed in the model that instead of appraising the risks of each individual securities. It is a better method of investment if trying to aggregate all the selected securities as a whole and next assess the collection of their risk according to their expect values and variance [1].

William Sharpe is the name that must be mentioned about when it comes to the Markowitz Model. William Forsyth Sharpe is an American economist who also won the 1990 Noble Prize in Economical sciences along with Harry Markowitz. One of his great contributions to the economic field is that he further developed portfolio analysis of Markowitz Model and created a more simplified model that are able to measure the risk and returns of stock with less variables involved [2]. Moreover, he also developed the Capital Asset Pricing Model (CAPM) [3], elaborating on the relationship between systematic risks and expected return. Sharpe's model construction and research lay a solid foundation for the assessment of risks and selections of maximum return portfolio.

Some previous empirical studies on Markowitz Model would be helpful to visualize the application of Markowitz Model in the real-world market. Radović, Radukić, and Njegomir put their target on Serbian capital market which they analyzed the statistics of 21 companies in the market. After calculating the risks and return with the setting of matrices of covariance, they concluded that

Markowitz Model is suitable for liquid shares, but when it comes to illiquid and insufficient securities, the Markowitz Model seems not applicable [4]. Apart from that, Sarker utilized Markowitz Model as the main measure to conduct a 5-year analysis on a sample size of 164 companies that listed on Dhaka Stock Exchange and ended with the conclusion that the Markowitz Model worked well in the Bangladesh Stock Market [5]. Ivanova and Dospatliev in their practical study of Bulgarian Stock market also proved that the optimal portfolio that formed by Markowitz Model is able to make the investors in Bulgaria earn a higher return than the investors using single security evaluation [6]. From the above practical studies on Markowitz Model, it is obvious that the optimizations on investor's portfolio brought by Markowitz Model cannot be neglected. The sample size of these studies further indicates that despite some insufficient securities existed in the real market, investors can still gain optimized portfolio by using the Markowitz Model. Besides from the comparison and contrast in the Markowitz Model and individual securities, some comparisons are also made between the full Markowitz Model and the single Index Model. In the article written by Varghese and Joseph, a notable distinction between these two models is that Markowitz Model is inclined to concentrate on "continued probabilistic growth and expansion," while Single Index Model tends to construct optimal portfolio by several key variables [7].

This article mainly focusses on calculating the optimization inputs for full Markowitz Model based on the stocks of ten selected companies, aiming to prove the feasibility of Markowitz Model under real-world conditions. By setting up the Model and formulas for the collected statistics, the major values like minimal risk, maximum Sharpe ratio, standard deviation with covariance and correlation, and portfolios under different constraints are contributed to the theory and potential application of Markowitz Model. The limitation of the model is also a section of this article as to manifest the rigorous logical structure.

2. Data & Methodology

2.1 Data

By referencing the previous empirical studies on the Markowitz Model, a twenty-year period of stock price data is preferred in this article, since the length of timescales with more complicated data load could be a potential factor that affect the accuracy of theoretical model implementation. The daily stock prices of ten large companies are selected, which are ADBE, IBM, SAP, BAC, C, WFC, TRV, LUK, ALK, and HA collected from Yahoo Finance. Each company will have a brief introduction in the following sectors. Besides from these ten company's stock prices, a market-capitalization-weighted index S&P 500 index will be incorporated into the calculation as an indicator to estimate the performance of selected securities. In addition, to fully present the Markowitz Model, a 1-month annual Fed Funds rate is used to represent the risk-free rate. Ten stock companies are categorized into three different types of stocks in order to demonstrate the diversification of the portfolio, which are Technology, Financial services, and Industries.

2.1.1 Technology

The first company is Adobe inc., which is also known as Adobe Systems Incorporated, is a company that designs software of printing, publishing, and graphics. Some of their products are very popular in both daily and work purposes of computer designation, such as Adobe Photoshop, Acrobat reader, and Adobe creative cloud. The second company in technology section is International Business Machines Corporations (IBM), a leading computer manufacturer in United States. Its business includes almost every component of computer, ranging from software to hardware production. The third technological stock is SAP SE, a multinational software company originated from Germany. It is one of the largest business software vendors which put their emphasis on business operation software and customer connections.

2.1.2 Financial Service

The next three stocks belong to financial service area rather than technology. Firstly, the Bank of America (BAC) is one of the largest banking and financial services corporations in the United States. Commercial banking, wealth management, and investment banking are the three main branches developed by BAC. Citigroup Incorporation (C) is another well-known financial service company headquartered in New York. It was the largest U.S. bank with over 3000 branch office worldwide, but subprime mortgage crisis in 2008 caused huge losses and stagnated its development. Wells Fargo Company is the third financial service corporation which supports all types of finance services like retail banking, investment banking, and personal finance. The Travelers Companies (TRV) is a financial institution that dedicated to insurance service.

2.1.3 Industry

Finally, the first industrial stock is Southwest Airlines (LUV), a civil aviation industry that developed rapidly in a few years and focuses on domestic routes in the United States. Alaska Air Group is the second industrial stock which also belongs to the category of aviation institution. Mainline, Region, and Horizon are the main air transportation services that ALK provides. Similar to ALK, Hawaiian Holdings is an aviation company that contains air travel and air cargo operations.

2.2 Model, Formula and Procedure

The objective of this article is trying to build an optimized portfolio of selected securities by using the Markowitz Model. According to Harry Markowitz, the approach is composed of three parts:

- Determine the minimal variance frontier so that the model enables the relations between risk and return.
- Construct the Capital Allocation Line (CAL) and identify the optimal risky portfolio by determine the tangential line with steepest slope of CAL.
- Select the suitable portfolio after blending the risk-free asset given risk-aversion [1].

Owing to the enormous datasets brought by 20-year stock price, the daily stock price may cause unexpected outliers when one tries to implement the Markowitz Model, thereby leading to unanticipated results. Hence, the daily stock prices should be aggregated into monthly stock prices in order to reduce bias and fluctuation. The first index that requires to be calculated is the Nominal Risk-Free Rate (NRFR) as it represents the yield of risk-free asset without the effect of inflation. It is very important for the later calculation in Sharpe ratio and efficient curve under different constraints.

$$NRFR_{month\ i} = NRFR_{month\ i-1} * (1 + FEDL01_{month\ i}/12) \quad (1)$$

Where month i-1 represents the nominal risk-free rate of previous month and FEDL01 represents the one-month federal funds rate.

The Markowitz Model has another name called mean-variance model, therefore mean and variance are the keys to analyze and construct the optimized portfolio and efficient frontiers. The equations (2) and (3) are the standard formula for calculating the expected return and variance when there are in assets.

$$E(r_p) = \sum_{i=1}^{n} w_i E(r_i) \quad (2)$$

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j) \quad (3)$$

Where w_i and w_j is equal to the weight of funds to be invested and $Cov(r_i, r_j)$ stands for the covariance of two risky assets i and j.

Furthermore, if one tries to find the point where has the highest slope between Capital Allocation Line and efficient frontier, the Sharpe Ratio would be a good measure to approach. Sharpe Ratio is a

mathematical expression that reflect the gain of excess return for certain amount of extra volatility in the risky asset. R_p stands for the return of portfolio and R_f stands for the risk-free rate.

$$Sharpe\ Ratio = \frac{R_p - R_f}{\sigma_p} \tag{4}$$

To further evaluate the risk and relations in various assets, creating a matrix of correlation is the most explicit way. If a higher correlation is found in the portfolio, it indicates higher risks and high volatility of the portfolio. The diversification effect brought by the Markowitz Model may not effective enough as well. On the other hand, a portfolio carries lower correlation means that the securities have less risks and tend to be discrete. A better performance of the portfolio and lower volatility would also be detected.

$$\Omega_{n\times n} = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_n^2 \end{pmatrix} \tag{5}$$

When the Markowitz Model is applied to solve practical situations, some constraints must be considered as well, because some accidental events may occur like tax-related constraints and socially responsible investment. The first optimization constraint aims to simulate the Regulation T by FINRA, which the customers can own 50% or more of the positions funded by the customer's account equity under the allowance of broker-dealers.

$$\sum_{i=1}^{11} |w_i| \leq 2 \tag{6}$$

The second constraint considers arbitrary "box" constraints on weights, which mainly come from client.

$$|w_i| \leq 1, for\ \forall i \tag{7}$$

The third constraint designed to be a "free" problem which basically shows the area of permissible portfolio and the efficient frontier under zero constraint. The fourth constraint tries to simulate the limitations existed in the U.S. mutual fund industry which no short positions should appear in a U.S. open-ended mutual fund.

$$|w_i| \geq 0, for\ \forall i \tag{8}$$

The fifth constraint takes into account the inclusion of the broad index in the portfolio, aiming to detect any positive or negative effect.

$$w_i = 0 \tag{9}$$

3. Results & Discussion

After performing the calculations and excel operations described in Sec. 2, the correlation between each selected stocks are shown in Table 1. Different categories of stocks are highlighted in different colors. The technology, financial services, and industry stocks are painted in yellow, orange, and blue respectively. The color difference also implies that the stocks that are in the same category tend to have more similarities, thereby having higher correlation and volatility rate. In contrast, if comparing the two stocks that belong to two different categories, the correlation is lower than the correlation of two stocks in the same category. Hence, the risks can be reduced to a lower level and the portfolio is

closer to the optimal portfolio. Although some fluctuations existed in different stocks' correlation, the overall statistics remains at a relatively low value, therefore the practical application of Markowitz Model in these ten selected stocks is still trustworthy.

Table 1. The Correlation Table

CORR	SPX	ADBE	IBM	SAP	BAC	C	WFC	TRV	LUV	ALK	HA
SPX	1.00	0.66	0.65	0.65	0.60	0.70	0.56	0.60	0.54	0.46	0.39
ADBE	0.66	1.00	0.46	0.53	0.42	0.46	0.30	0.45	0.39	0.23	0.18
IBM	0.65	0.46	1.00	0.59	0.31	0.42	0.27	0.38	0.35	0.36	0.25
SAP	0.65	0.53	0.59	1.00	0.33	0.43	0.30	0.38	0.32	0.28	0.14
BAC	0.60	0.42	0.31	0.33	1.00	0.83	0.76	0.39	0.43	0.28	0.34
C	0.70	0.46	0.42	0.43	0.83	1.00	0.70	0.51	0.43	0.30	0.34
WFC	0.56	0.30	0.27	0.30	0.76	0.70	1.00	0.34	0.41	0.35	0.36
TRV	0.60	0.45	0.38	0.38	0.39	0.51	0.34	1.00	0.41	0.36	0.24
LUV	0.54	0.39	0.35	0.32	0.43	0.43	0.41	0.41	1.00	0.52	0.42
ALK	0.46	0.23	0.36	0.28	0.28	0.30	0.35	0.36	0.52	1.00	0.40
HA	0.39	0.18	0.25	0.14	0.34	0.34	0.36	0.24	0.42	0.40	1.00

Fig. 1 shows all the individual assets that generated by the formula of expected return, standard deviation, ranging from -25% to 40%. According to the results, most of the individual assets are concentrated at the area where standard deviation is greater than 0.4. The next step will be setting up the Capital Allocation Line and minimal variance frontier (No constraint or constraint 3) with respect to the values of these individual assets.

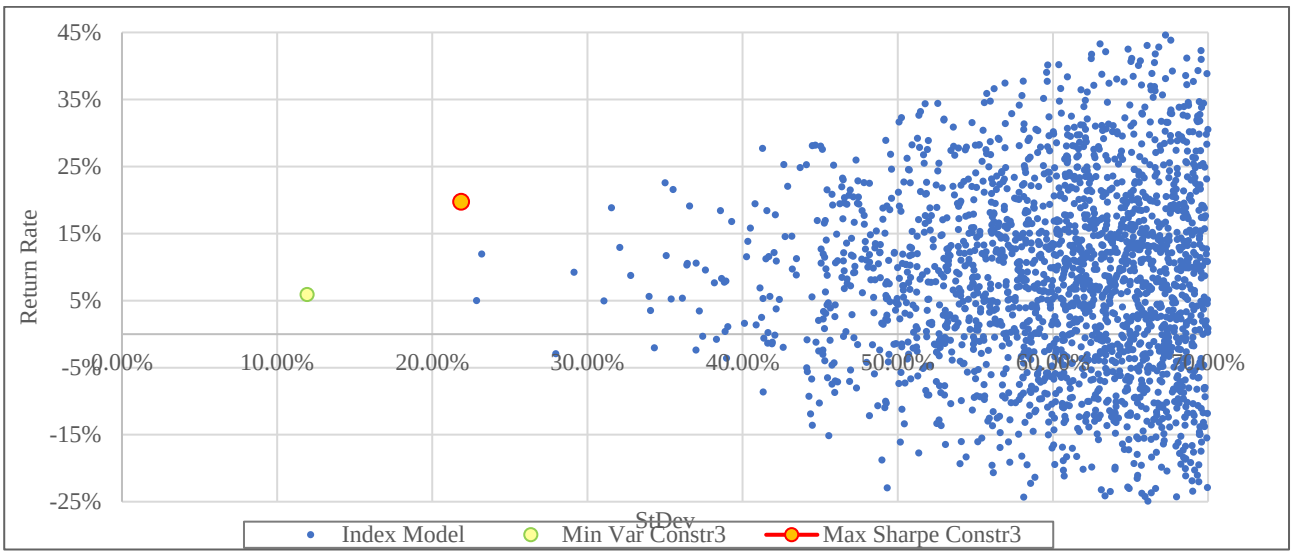

Figure 1. Markowitz Model with Individual Assets

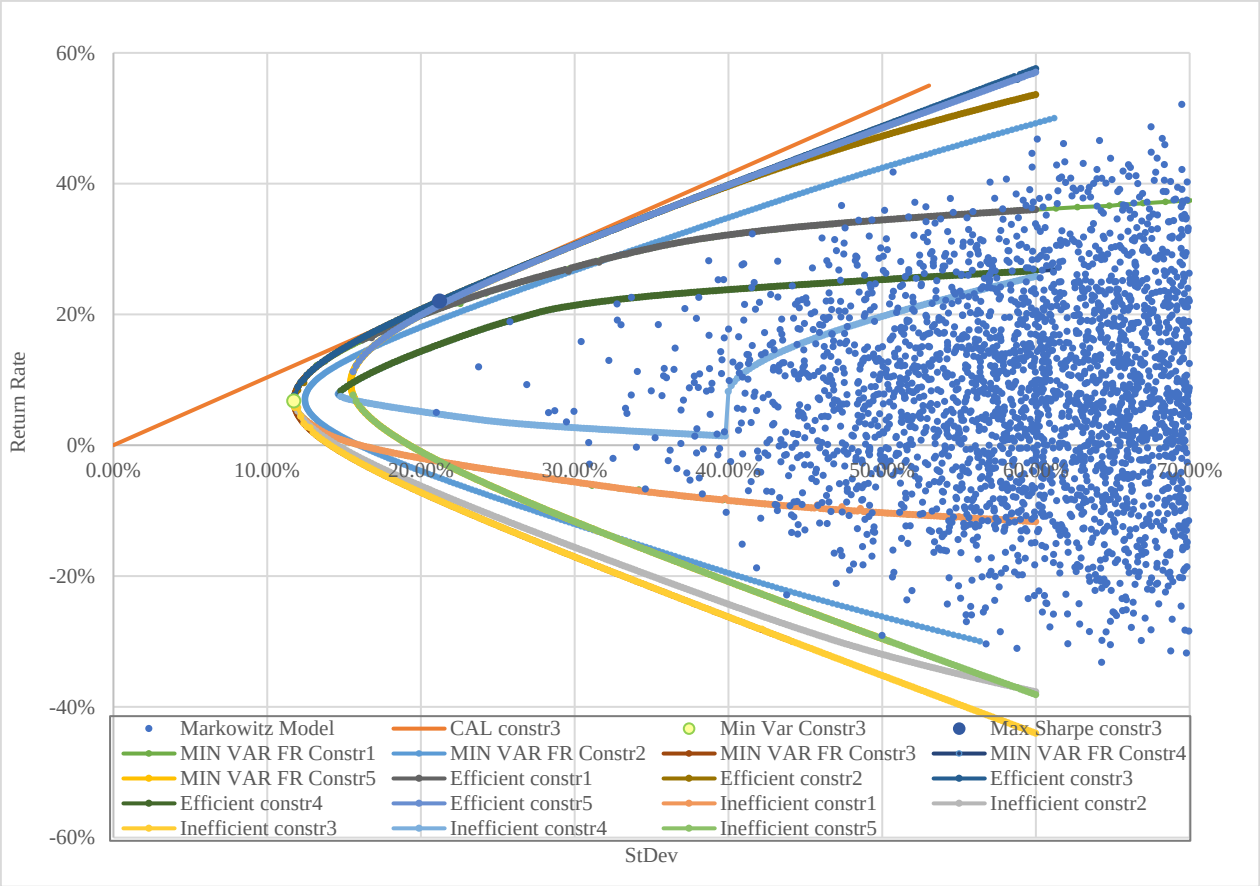

Figure 2. Markowitz Model Under All Five Constraints

The minimal variance frontiers and permissible portfolio under constraint 3 is the top priority to consider as no extra limitations are implemented on the statistics. The results can be observed from part of Fig. 2. The optimal risky portfolio is also visible by the max Sharpe Ratio point which is the intersection of CAL and minimal variance curve at the highest reward to volatility ratio (Steepest slope). Moreover, the minimal variance frontier of risky assets is divided into two parts, i.e., efficient frontier and inefficient frontier. The upper blue curve is the efficient frontier since they provide best risk-return combinations, whereas the lower yellow curve is the inefficient frontier because of their inability to give the best risk-return combinations. The point that separates them is called global minimum variance portfolio.

Table 2. Minimal Variance and Max Sharpe Ratio Under Different Constraints

MM (Constr1):	**SPX**	**ADBE**	**IBM**	**SAP**	**BAC**	**C**	**WFC**	**TRV**	**LUV**	**ALK**	**HA**	**Return**	**StDev**	**Sharpe**
MinVar	1.11	-0.10	0.05	-0.10	0.00	-0.23	0.14	0.19	0.00	-0.05	-0.03	6.8%	11.7%	57.6%
MaxSharpe	0.40	0.28	0.00	0.00	0.17	-0.48	0.20	0.30	-0.02	0.08	0.08	17.6%	17.6%	99.7%
MM (Constr2):	**SPX**	**ADBE**	**IBM**	**SAP**	**BAC**	**C**	**WFC**	**TRV**	**LUV**	**ALK**	**HA**	**Return**	**StDev**	**Sharpe**
MinVar	1.00	-0.09	0.08	-0.09	0.01	-0.23	0.16	0.23	0.00	-0.05	-0.03	1.9%	11.8%	59.5%
MaxSharpe	0.51	0.35	-0.22	0.07	0.30	-0.66	0.21	0.35	-0.15	0.13	0.11	-10.4%	21.2%	103.7%
MM (Constr3):	**SPX**	**ADBE**	**IBM**	**SAP**	**BAC**	**C**	**WFC**	**TRV**	**LUV**	**ALK**	**HA**	**Return**	**StDev**	**Sharpe**
MinVar	1.11	-0.10	0.05	-0.10	0.01	-0.23	0.14	0.20	0.00	-0.05	-0.03	6.7%	11.7%	57.4%
MaxSharpe	0.51	0.35	-0.22	0.07	0.30	-0.66	0.21	0.35	-0.15	0.13	0.11	22.0%	21.2%	103.7%
MM (Constr4):	**SPX**	**ADBE**	**IBM**	**SAP**	**BAC**	**C**	**WFC**	**TRV**	**LUV**	**ALK**	**HA**	**Return**	**StDev**	**Sharpe**
MinVar	0.84	0.00	0.00	0.00	0.00	0.00	0.00	0.16	0.00	0.00	0.00	3.5%	14.6%	53.7%
MaxSharpe	0.00	0.50	0.00	0.00	0.00	0.00	0.00	0.18	0.00	0.19	0.13	-3.0%	24.7%	73.9%
MM (Constr5):	**SPX**	**ADBE**	**IBM**	**SAP**	**BAC**	**C**	**WFC**	**TRV**	**LUV**	**ALK**	**HA**	**Return**	**StDev**	**Sharpe**
MinVar	0.00	0.04	0.34	-0.02	0.01	-0.25	0.35	0.53	0.05	-0.03	-0.02	-7.6%	15.4%	61.0%
MaxSharpe	0.00	0.51	-0.19	0.14	0.37	-0.77	0.29	0.50	-0.16	0.18	0.14	-17.0%	25.9%	102.3%

Shifting from perfect state to more realistic situations, the remaining four constraints are calculated and added into the Markowitz Model graph. Through applying the variance and excess return formula and using some auxiliary tools including Excel solver and solver table, the minimum variance and max Sharpe ratio for the ten selected stocks are presented in Table 2. The data varies significantly from one constraint to another. For example, constraint 4 seems to have the lowest Sharpe ratio and highly deviated values in both max Sharpe and minimal variance, which may indicate the highest risks and potential failure to optimized portfolio. The minimal variance segment of Constraint 1 reveals the highest resemblance to the minimal variance of constraint 3, implying that the limitations of constraint 1 on minimum variance is negligible and have a certain degree of influence on its max Sharpe ratio. On the contrary, the max Sharpe ratio of constraint 2 exhibits highly resemblance to that of constraint 3, whereas minor difference is discovered in minimum variance, forecasting that the restrictions of constraint 2 have inappreciable impact on max Sharpe ratio, but somewhat notable in minimum variance. Lastly, constraint 5 owns the highest return and standard deviation among all five constraints, plus the deviation of minimum variance and max Sharpe ratio from constraint 3 is also relatively high. This may lead to the conclusion that constraint 5 is a strict constraint that can produce high bias, but not as extremely as constraint 4 due to its allowance of short positions.

4. Limitations

Nowadays, the Markowitz Model has been a common tool that applied widely in the theoretical finance and investments. The primary reason that the Markowitz Model can only be used in theoretical area of finance is because it still cannot be considered as a convincing approach to the real-world problems. To be more specific, the portfolio selection in the Markowitz Model tends to be a one-off decision made in an extremely short period of time after examinations and predictions of the available securities, and disregard the later reallocation and review of portfolio [8]. More importantly, the pivotal prerequisite of this portfolio analysis model is assuming that the whole market is in an efficient condition and investors are completely rational and risk-averse. Nevertheless, investors in real world cannot hold rational views on their portfolio and market, thereby leaning to irrational decisions [9]. Furthermore, the theoretical bases of the Markowitz Model were challenged by Bowen as well. Bowen claimed that there are huge gaps between the practices and theory and pointed out that few firms require such complicated budgeting measures. He also holds the perspective that the risk, i.e., fundamental concept of Markowitz Model is not accurate enough by calculating of variance [10]. Despite its limitations in practical applications, it can give people a general sense of direction in the industry of finance and investment.

5. Conclusion

In summary, this article dedicates to the applications of the real-world situations and statistics in the well-known portfolio optimization model, i.e., the Markowitz Model. By selecting ten stocks from American stock market and aggregating the enormous statistics of individual assets, the basis of the model analysis is framed. Apart from that, the formulas of variance, excess return and Shape ratio are the mediums to calculate the minimum variance frontier and max Sharpe ratio. At last, through the implementations of five different constraints and claims of the defects inside the ideal assumptions of the Markowitz Model, the results fully reveal the feasibility and limitations of the optimization model. It should be noted that the Markowitz Model is not an omnipotent model for any real-world cases, which means it does provide bias and outliers in application. To analyze more complicated cases, more models and advanced knowledge are required beyond the Markowitz Model. Therefore, in the future, a more advanced optimized model may be developed based on the Markowitz Model. The practical analysis of optimization portfolio theory offers a valuable advice to the investors that the Markowitz Model, the evaluation of collective risks and various portfolios from aggregated

securities, is a better choice than only focusing on the risk and return of individual security under most real-world cases.

References

[1] Markowitz Harry. Portfolio Selection. The Journal of Finance, 1952, 7 (1): 77 – 91.

[2] Sharpe W F. A Simplified Model for Portfolio Analysis. Management Science, 1963, 9 (2): 277 – 293.

[3] Sharpe W F. Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. The Journal of Finance, 1964, 19 (3): 425 – 442.

[4] Radović Milica, et al. The Application of the Markowitz's Model in Efficient Portfolio Forming on the Capital Market in the Republic of Serbia. Economic Themes, 2018, 56 (1): 17 – 34.

[5] Mokta Rani Sarker. Markowitz Portfolio Model: Evidence from Dhaka Stock Exchange in Bangladesh. IOSR Journal of Business and Management, 2013, 8 (6): 68 – 73.

[6] Ivanova M, and Dospatliev L. Application of Markowitz Portfolio Optimization on Bulgarian Stock Market from 2013 to 2016. International Journal of Pure and Apllied Mathematics, 2018, 117, 2,

[7] Varghese J and Joseph A. A Comparative Study on Markowitz Mean-Variance Model and Sharpe's Single Index Model in the Context of Portfolio Investment. 2018, 3: 36 - 41.

[8] Cohen K J, and Jerry A. P. An Empirical Evaluation of Alternative Portfolio-Selection Models. The Journal of Business, 1967: 40 (2) 166.

[9] Adams Jacob. What Are the Benefits, Cons, and Limitations of Modern Portfolio Theory? Realized 1031 Exchange Marketplace, Realized, 28 Oct. 2021, Retrieved from: www.realized1031.com/blog/what-are-the-benefits-cons-and-limitations-of-modern-portfolio-theory.

[10] Bowen P. A. A Hypothesis: Portfolio Theory Is Elegant but Useless. Investment Analysts Journal, 1984, 13 (24): 17 – 21.