Portfolios Optimization under Constraints

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Abstract. This article selects 10 companies in the financial sector, energy sector and consumption sector, as well as SPX500 index. This paper uses two models, not only the Markowitz model but also the index model, to calculate the correlation coefficient matrix, minimum variance, maximum Sharpe ratio, capital allocation line and so on to analyze the return rate and volatility of 10 specific companies. Four limitations were calculated for Markowitz model and Index model respectively and the two models were compared under the same constraints. Because common financial constraints and specific industries are rarely noticed in reality, the results of this paper reflect the following three aspects: First, in order to strike a balance between risk and return, SPX is an investment worth considering due to its high correlation coefficient; the second is that for certain investors with added constraints, the capital allocation line performs relatively poorly, as does the minimum variance boundary. Thirdly, because the Markowitz model uses stock's covariance while the beta and alpha of stocks are components that the index model uses to construct a portfolio, the results show that under certain risk conditions, Markowitz model is inferior to index model in pursuing maximum return and minimum risk under certain return conditions.

Keywords: Markowitz model; Index Model; covariance; constraints.

1. Introduction

With the rapid development of the world economy and frequent occurrence of financial crisis and market fluctuations, the capital market has made great progress after the reform and opening up, but it is still not perfect and mature, which makes investors face more and more complicated theoretical and practical problems of financial investment decisions. Markowitz, an American economist, first proposed the mean-variance (MV) model in 1952, which laid the foundation for pursuing the maximum return of investment portfolio under the condition of low risk. Individual and institutional investors are the main participants in the market. Compared with institutional investors, although most individuals do not have a high risk tolerance, they have the idea of higher returns. Therefore, it is particularly important to diversify the risk through investment diversification, that is, investment portfolio. As it is pointed out, in addition to systemic risk -- a type of market risk that all sectors and stocks may face, it is a non-diversifiable risk [1]; for non-systemic risk, investing in more than one stock can have the diversification benefit of reducing portfolio risk. There are even some portfolios that don't include high-return companies because they're risky and SIM is useful in creating an optimal portfolio by diversifying almost all non-systemic risk [2]. Even before 1952, Shakespeare had put diversification in perspective in The Merchant of Venice, but Markowitz was the first to publicly show the advantages of diversification -- the only way to fulfill free profit of finance [3]. Caldeira, Moura and Santos pointed out that the expected return and covariance of bonds can be derived by using the dynamic factor model, which is a new method to study portfolio optimization in the US bond market. The research direction of this paper involves mean-variance model, and the expression in the dynamic factor model of yield curve can be used for mean-variance optimization, and the optimal mean-variance bond portfolio is obtained [4]. Although China's A-share market was established relatively late, in 1991, it has developed into the world's second largest stock market [5]. Between 2006 and 2019, the number of funds increased 17-fold. Institutional investors dominate trading in the US, with retail investors accounting for less than 20 per cent of volumes; by contrast, more than 80 per cent of trading volume in China's stock market can be attributed to retail investors. This may be due to the different maturity and saturation degree of institutional investors in the stock markets of China and the United States [5]. Chao, Tao, Zeng claimed that the calculation of historical

data will use the theory of Markowitz conceptual model of investment, including the calculation of weekly average interest rate and covariance matrix and found it in the domestic A-share market [6]. Although there are shortcomings, Markowitz's portfolio concept has specific application value.

This article focuses on several representative sectors of the market, namely technology, energy and consumption, and analyzes them under different constraints faced by different investors. First, take the stock prices of 10 companies from the U.S. S&P 500 index and from the technology, energy and consumer industries between May 11, 2001 and May 12, 2021. Then, the obtained financial data is processed and the redundant data is removed to facilitate the subsequent processing. Then, in order to find the correlation coefficients of enterprises in the same industry and enterprises in different industries, the correlation coefficients between 11 kinds of underlying assets are calculated and the correlation coefficients of enterprises in the same industry are higher than those of enterprises in different industries. In addition, no matter what kind of assets, and the SPX500 index has a high correlation coefficient. Finally, the minimum variance boundary and capital allocation line under each constraint condition are obtained by using the solution table, which is actually to increase the constraints, but it is found that these constraints will reduce the effectiveness of diversification.

After the introduction, the following structure follows. Section 2 provides the source of the data and the presentation of the results. The third part is the method of data processing. Section 4 visualizes the data and Section 5 summarizes the article.

2. Data and Methodology

2.1 Data

Yahoo finance (https://finance.yahoo.com) is the data acquisition of this article. This article takes a look of the closing prices at the SPX500 index and ten representative companies from three sectors: financials, energy and consumer from May 1, 2001 to May 1, 2021. Specifically, MCD, PEP, KO, IMO, XOM, CVX, MSFT, ORCL, AKAM, QCOM. The companies mentioned above were chosen because they are representative of specific industries and have been listed for more than 20 years. These 10 stocks have performed steadily over the past 20 years. Then, in order to reduce the non-Gaussian effect, a total of 2662 data were collected from daily observations and aggregated into monthly observations with 240 final data. As shown in Table 1, AAR (annualized average return), ASD (annualized standard deviation), and beta, alpha and RSD (residual standard deviation) are calculated. As shown in Table 2, for the Markowitz model, the correlation coefficient matrix of each stock is calculated.

Table 1. Descriptive statistics of the selected stocks

	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD
AAR	5.762%	11.627%	28.594%	9.561%	10.994%	5.227%	2.217%	10.297%	4.449%	5.835%	11.202%
ASD	13.251%	33.217%	63.281%	27.397%	23.386%	22.238%	20.711%	30.728%	16.146%	15.093%	18.399%
β	1.0000	0.7042	1.04193	0.70606	0.51281	0.48499	0.49945	0.70741	0.31142	0.36498	0.30925
α	0.000%	7.569%	22.591%	5.493%	8.039%	2.433%	-0.661%	6.221%	2.655%	3.732%	9.421%
RSD	0.000%	31.880%	61.757%	25.750%	22.377%	21.290%	19.625%	29.263%	15.610%	14.297%	17.937%

Table 2. Correlation matrix

correlations	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD
SPX	1	0.2809	0.2182	0.3415	0.2906	0.2890	0.3196	0.3051	0.2556	0.3204	0.2227
QCOM	0.2809	1	0.2782	0.2989	0.3758	0.2356	0.2383	0.2750	0.1956	0.2653	0.2563
AKAM	0.2182	0.2782	1	0.2578	0.2609	0.1115	0.0641	0.1252	0.0774	0.1067	0.2726
ORCL	0.3415	0.2989	0.2578	1	0.4749	0.2905	0.3186	0.2440	0.0846	0.2095	0.1764
MSFT	0.2906	0.3758	0.2609	0.4749	1	0.3507	0.3093	0.2502	0.2895	0.3348	0.3648
CVX	0.2890	0.2356	0.1115	0.2905	0.3507	1	0.8277	0.7327	0.3918	0.2759	0.3936
XOM	0.3196	0.2383	0.0641	0.3186	0.3093	0.8277	1	0.6972	0.3340	0.2483	0.3476
IMO	0.3051	0.2750	0.1252	0.2440	0.2502	0.7327	0.6972	1	0.3025	0.1844	0.2751
KO	0.2556	0.1956	0.0774	0.0846	0.2895	0.3918	0.3340	0.3025	1	0.5798	0.4982
PEP	0.3204	0.2653	0.1067	0.2095	0.3348	0.2759	0.2483	0.1844	0.5798	1	0.4836
MCD	0.2227	0.2563	0.2726	0.1764	0.3648	0.3936	0.3476	0.2751	0.4982	0.4836	1

As shown in the table 1, we can draw some conclusions that although AKAM Company has the highest annualized average return, it also has the largest annualized standard deviation, which is not a sound portfolio choice. XOM had the lowest annualized average return. It is worth noting that although MCD ranks third in annualized average return, its annualized standard deviation is much lower than the top two companies, AKAM and QCOM, with higher return, which is a sound portfolio choice since MCD's higher annual return and lower volatility. We can also see that only XOM has an alpha value below 0. From the table 2 correlation matrix, the correlation coefficient between XOM and CVX is the largest, and the correlation coefficient between XOM and AKAM is the smallest.

2.2 Method

In Markowitz's "Portfolio Selection" published in the Financial Journal in 1952, the concept of MPT fundamentally changed the way investments were made [7]. Gradually, the Markowitz model has formed a preliminary theoretical framework for analyzing the relationship between risk and return. The portfolio constructed by Markowitz model gives the portfolio with the highest return under the given risk level, namely an efficient frontier [8]. Statistical analysis is used to measure risk, and mathematical programming is used to efficiently select the assets in a portfolio. An efficient portfolio that can generate the lowest risk at a given level of return is called an efficient frontier, or the highest return at a given level of risk is called an efficient frontier [6]. Suppose there are n risky assets, assets' return rate are r_1, r_2, ..., r_n respectively and investor in risk assets' allocation ratio are W_1, W_2, ..., W_n respectively. Then for Markowitz model, the portfolio's return rate is:

$$\sum_{i=1}^{n} w_i = 1 \tag{1}$$

$$r_p = \sum_{i=1}^{n} r_i * w_i \tag{2}$$

So the expected rate of return of Markowitz model is:

$$E(r_p) = \sum_{i=1}^{n} E(r_i) * w_i \tag{3}$$

The variance is:

$$Var(r_p) = \sqrt{\sum_{i=1}^{n}(w_i)\text{^}2\, Var(r_i) + \sum_{i,j}^{n} w_i * w_j * C_{ij}} \quad (4)$$

where C_{ij} stands for the correlation between stock i and stock j. So the expected rate of return of Index model is:

$$r_I = \sum_i w_i * (\alpha_i + \beta_i * r_{Market} + R_i) \quad (5)$$

where the α coefficient (α) represents the return over the risk-free rate, $\beta_i * r_{Market}$ term represents the stock return modified by the beta i of the stock due to market movements [9].

3. Result

In this paper, four constraints are considered. Detailed description and the related results are shown in the following Tables 3-6.

Constraint 1: $\sum_{i=1}^{11} |w_i| \leq 2$

Constraint1 is a representation of a credit guarantee. The essence is to promise that there is enough capital for the securities company to buy and sell securities on behalf of customers, and to bear the possible adverse consequences within the margin. The results for the two models under constraint 1 is shown below.

Table 3. Weights under constraint 1

MM	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MV	30.95%	-2.194%	-1.05%	5.34%	-0.38%	-8.70%	19.56%	-3.86%	20.96%	30.52%	8.857%	7.232%	12.28%	58.89%
MS	-40.93%	6.044%	6.23%	14.92%	20.88%	5.77%	-9.07%	11.51%	5.12%	28.98%	50.55%	14.59%	16.11%	90.54%
IM	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MV	30.954%	-2.194%	-1.05%	5.335%	-0.376%	-8.70%	19.561%	-3.86%	20.96%	30.520%	8.857%	7.232%	12.28%	58.89%
MS	-48.43%	4.393%	6.201%	6.532%	21.74%	4.441%	-1.562%	3.814%	20.48%	33.528%	48.871%	13.64%	14.55%	93.73%

Constraint 2: $|w_i| \leq 1$, for $\forall i$

Constraint 2: Each trader has his own preferences and styles, and Constraint 2 shows what a portfolio would look like without an index. The results for the two models under constraint 2 is shown below.

Table 4. Weights under constraint 2

MM	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MV	30.954%	-2.194%	-1.05%	5.335%	-0.376%	-8.70%	19.561%	-3.86%	20.96%	30.520%	8.857%	7.232%	12.28%	58.89%
MS	-100.0%	12.227%	8.396%	24.181%	30.41%	26.11%	-30.03%	18.09%	9.452%	38.510%	62.645%	17.90%	19.19%	93.28%
IM	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MV	12.912%	-4.215%	-2.50%	-0.524%	-0.086%	3.300%	9.097%	-1.38%	29.76%	36.724%	16.916%	7.543%	11.36%	66.37%
MS	-93.84%	8.183%	7.929%	10.731%	29.77%	10.16%	-3.676%	7.245%	25.43%	40.305%	57.762%	15.56%	16.33%	95.32%

Constraint 3: $w_i \geq 0$, for $\forall i$

Constraint 3: Because there are few types of financial derivatives and the financial system is not sound enough - take China's A-share market as an example. This article sets Constraint3 for them -- shorting is not allowed. The results for the two models under constraint 3 is shown below.

Table 5. Weights under constraint 3

MM	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MV	19.75%	0.000%	0.00%	6.28%	0.00%	0.0%	11.18%	0.00%	21%	33.46%	8.34%	8.03%	12.4%	64.7%
MS	0.000%	2.107%	5.34%	8.38%	14.83%	0.0%	0.000%	5.30%	0.%	19.71%	44.35%	12.8%	15.1%	84.7%
IM	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MV	12.912%	-4.215%	-2.50%	-0.524%	-0.086%	3.30%	9.097%	-1.38%	29.76%	36.72%	16.92%	7.543%	11.364%	66.37%
MS	-93.84%	8.183%	7.929%	10.731%	29.774%	10.16%	-3.68%	7.25%	25.43%	40.31%	57.76%	15.56%	16.322%	95.32%

Constraint 4: $w_1 = 0$. The results for the two models under constraint 4 is shown below.

Table 6. Weights under constraint 4

MM	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MV	0.000%	0.806%	-0.07%	9.178%	3.72%	-5.11%	21.75%	-2.79%	24.81%	36.093%	11.600%	8.062%	12.457%	65.52%
MS	0.000%	2.406%	4.816%	11.569%	16.17%	13.02%	-32.66%	13.30%	-0.895%	21.688%	50.589%	14.38%	16.537%	86.98%
IM	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MV	0.000%	-3.353%	-2.25%	0.49%	1.60%	4.92%	10.62%	-0.53%	31.37%	38.66%	18.48%	7.88%	11.40%	69.15%
MS	0.000%	1.721%	5.79%	3.25%	16.92%	-1.37%	13.78%	1.01%	14.39%	26.73%	45.34%	12.91%	14.59%	88.52%

Since two methods are based on the same 10 stocks and SPX index, which will gives the same basic information about annualized average return, annualized standard deviation and correlation. However, the covariance of stocks are used to construct the Markowitz portfolio; beta and alpha of stocks are used to construct the Index portfolio. According to them, calculation required by the Markowitz model is more than the Index model, but also Markowitz model's optimal result is worse than the index model because under each constraint Index model's MaxSharpe is greater than Markowitz model with a lower standard deviation. Selected ADBE (Adobe), ALK (Alaska Air Group Inc), BAC (Bank of America), C (Citigroup), IBM (International Business Machines) and LUV (Southwest Airlines), when the risk increased by the same amount, the Index model always returns less than the Markowitz model [10]. Therefore, the advantages and disadvantages of Markowitz model and index model can be obtained only after we select the investment portfolio and carry out corresponding calculations.

4. Conclusion

Most portfolios study general market conditions and specific industries. The purpose of this article is to analyze investments in the technology, energy and consumer sectors and promote smarter investments. The advantages and disadvantages of Markowitz model and Sharpe single index model are shown in this paper. In order to observe the real financial market, the influence of four constraints with practical significance on the investment portfolio is calculated respectively. Different constraints on the same model and different models under the same constraint are analyzed. The Index model outperforms the Markowitz model in most cases.

Only a few representative companies were selected from the three industries, which is not very representative enough. Too many assumptions in the mean-variance model may be inconsistent with the reality of financial conditions.

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