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# Term Structure Model of Interest Rates

# -- A Literature Review

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## Abstract

**Term structure modeling has enjoyed rapid growth during the last three decades. Given a large number of existing term structure models and a vast array of issues in the field, we attempt to provide a general overview of the most popular term structure of interest rate models. In order to understand different features of each model we classify by means of general characteristics from single-factor to multi-factor and forward rate based models. Each of these existing term structure models has its own advantages and disadvantages. We also highlight the recent advocated models in the literature: the Nelson-Siegel model, the affine and the quadratic arbitrage-free model.**

## Keywords

**Term Structure Modeling; Nelson-Siegel Model; The Affine Term Structure Model.**

## 1. Introduction

The term structure of interest rates, also known as the yield curve, plays a central role both theoretically and practically in the economy. It gives the relationship between the yield on an investment and the term maturity of the investment. Yield curve modeling literature has at the origin the need of explaining interest rate behavior. Both short term and long term interest rates have an important role in financial markets, for different reasons and purposes. They are used for the price of borrowing or lending money; they are needed to price bonds and to price derivatives on bonds and other fixed income instruments. Thus understanding and modeling the term structure of interest rates has been one of the most challenging topics of financial research.

The characteristic features of interest rate models can be generally categorized into eight types: continuous or discrete models, single or multi-factor models, fitted (to the initial term structure) or non-fitted models and arbitrage-free or equilibrium model.

In order to keep the scope manageable, the aim of this paper is to provide an analysis of the most popular term structure models of interest rates that are applicable to the default-free zero-coupon bonds. We propose these models in a common framework and explain their merits and drawbacks from an overview perspective.

This paper is organized as follows: section 2 introduces the definitions and notations will be used throughout the paper; section 3 reviews simple factor interest rate models in the literature with both time-invariant and time varying parameters; section 4 considers the extension of the single factor model to multi-factor model. Section 5 reviews the forward rate based Heath, Jarrow and Morton (1992) model which models the entire term structure and provides a richer volatility pattern for predicting and controlling future volatilities. We highlight three empirically more advanced term structure models for bond yields in section 6 and conclude in section 7.

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## 2. Definitions and Notations

The models will generally be set up in a filtered probability space $(\Omega, F_t, \mathrm{P})$, where Ω is the sample space, $F_t$ is the sigma-field generated by a standard Brownian motion W(t). P indicates the historical (physical) probability measure on the sample space Ω.

We denote $P(t,T)$ as the price at time t of a default-free zero-coupon bond with principal one dollar maturing at time T. It follows that $P(T,T)=1$. At time t, the yield to maturity $y(t,T)$ of the zero-coupon bond is the continuously compounded rate of return that causes the bond price to rise to one at time T. Yields are solved by $y(t,T) = -\frac{\ln P(t,T)}{T-t}$.

For a fixed time t, the shape of the yield $y(t,T)$ as T increases determines the term structure of interest rates. Since we only work with the zero-coupon bonds, the yield curve is the same as the term structure of interest rates.

The instantaneous risk-free rate also called short rate/short term rate is denoted as $r(t)$ and $r(t) = \lim_{t \to T} y(t,T)$.

Define $f(t,T_1,T_2)$ as the forward rate at time t for the period between time $T_1$ and $T_2$. The relationship between forward rate and zero-coupon bond is given by:

$$f(t,T_1,T_2) = \frac{\ln P(t,T_1) - \ln P(t,T_2)}{T_2 - T_1} \tag{1}$$

The instantaneous forward rate is the rate that one contracts at time t for a loan starting at time T for an instantaneous period of time $f(t,T) \equiv f(t,T,T)$.

The bond price can be defined in terms of forward rate as:

$$P(t,T) = e^{-\int_t^T f(t,s)du} \tag{2}$$

The relationship between short rate and forward rate is given by $r(t) = f(t,t)$, so the short rate is actually a specific forward rate.

## 3. Single Factor Models

Factor models assume that the term structure of interest rates is driven by a set of state variables or factors. A principle component analysis[40] can be used to decompose the motion of the interest rate term structure into three independent factors: shift, twist and butterfly of the term structure (Wilson, 1994). As the first principle component (shift) explains a large fraction of the yield curve movement, it is tempting to reduce the problem to a single factor model.

Single factor models assume that all information about the term structure at any point in time can be summarized by one single factor – the short rate $r(t)$. As a consequence, only the short rate and time to maturity will affect the price of the zero-coupon bonds. There are two basic methodologies for pricing interest rate contingent claims in a single factor framework, the partial differential equation and the martingale approach. The former creates an instantaneous risk-free portfolio to obtain a second order partial differential equation that interest rate contingent claim must satisfy. The latter proposed by Harrison and Kreps (1979) and extended by Heath, Jarrow and Morton (1992) uses the result that in a complete market, in the absence

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of arbitrage, there exists an equivalent martingale measure under which asset prices can be computed as an expectation. These two approaches are equivalent by the theorem of Feynman-Kac.

There are three particular versions of the single factor models: the affine class models, the Gaussian models and the lognormal models.

The affine models named by Duffie and Kan (1996) have the following exponential affine form of the zero-coupon bond price:

$$P(t,T)=\exp\left[a(t,T)r(t)+b(t,T)\right] \quad (3)$$

where $a(t,T)$ and $b(t,T)$ are deterministic functions that can be calculated via Riccati ordinary differential equation (obtained from the partial differential equation of the bond pricing). The term structure of interest rates is an affine function of the short rate:

$$y(t,T)=\frac{-a(t,T)}{T-t}r(t)+\frac{b(t,T)}{T-t} \quad (4)$$

If under the risk-neutral probability, the mean and volatility are affine in $r(t)$, then we say that the model has an affine version.

A short term interest rate model is said to be Gaussian if it can be written as the following linear stochastic differential equation (SDE):

$$dr(t)=\mu_r(t,r(t))dt+\sigma_r(t,r(t))dW(t) \quad (5)$$

where $\mu_r$ and $\sigma_r$ are the drift and the volatility/standard deviation of the short rate, respectively. Gaussian model is a particular class of affine models, and $r(t)$ is normally distributed.

A short term interest rate model is said to be lognormal if and only if $\ln r(t)$ is Gaussian. The advantage of lognormal models over Gaussian is that by definition, lognormal rate models cannot generate negative interest rates. However, they generally lack analytical tractability.

So far we have discussed the versions of the single factor models, now we turn to some examples of the single factor models in the literature.

## 3.1. One-factor Time Invariant/ Equilibrium Models

### 3.1.1. Merton (1973)

Merton was the first to propose a general stochastic process as a model for the short rate. Under the historical probability measure P, the short rate has the following SDE:

$$dr(t)=\mu dt+\sigma dW(t) \quad (6)$$

where $\mu$ and $\sigma$ are constant.

The short rate is solved by $r(t)=r(s)+\mu t+\sigma\int_s^t dW(s)$ for any $t\geq s$.

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Given the set of information at time s, the short term rate $r(t)$ is normally distributed with mean $r(s)+\mu(t-s)$ and variance $(t-s)\sigma^2$. The unboundedness of the first and second moment of the distribution allows the rate to become negative or infinite.

#### 3.1.2. Vasicek (1977)

Vasicek proposed to model the short term interest rate as a Gaussian Ornstein-Uhlenbeck (mean-reverting) process:

$$dr(t)=\kappa\left(\theta-r(t)\right)dt+\sigma dW(t) \tag{7}$$

where $\kappa,\theta$ and $\sigma$ are constants. This model incorporates mean reversion. The short rate is pulled to a level $\theta$ at the mean reversion rate $\kappa$.

The explicit solution to the SDE in (7) gives us the short term rate $r(t)=\theta+\left(r(s)-\theta\right)e^{-\kappa(t-s)}+\sigma\int_s^t e^{-\kappa(t-s)}dW(u)$ for any $t\geq s$.

Given the set of information at time s, the short term rate $r(t)$ is normally distributed with mean $\theta+\left(r(s)-\theta\right)e^{-\kappa(t-s)}$ and variance $\frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa(t-s)}\right)$. Again, the short term interest rate can become negative which is the significant drawback.

#### 3.1.3. Cox, Ingersoll and Ross (1985)

Cox, Ingersoll and Ross have developed an alternative model where rates are always non-negative. The short term rate satisfies the following SDE:

$$dr(t)=\kappa\left(\theta-r(t)\right)dt+\sigma\sqrt{r(t)}dW(t) \tag{8}$$

where $\kappa,\theta$ and $\sigma$ are constants. This model has the same mean-reverting drift as the Vasicek model, but the variance of the change in the short rate in a short period of time is proportional to the short rate r rather than constant. This means that, as the short term interest rate increases, its standard deviation increases.

The positive short term rate is obtained by solving the SDE in (8), $r(t)=\theta+\left(r(s)-\theta\right)e^{-\kappa(t-s)}+\sigma e^{-\kappa(t-s)}\int_s^t e^{\kappa(u-s)}\sqrt{r(u)}dW(u)$ for any $t\geq s$.

Given the set of information at time s, the short term rate $r(t)$ is distributed as a non-central chi-squared (Feller, 1951).

The one-factor time-invariant/equilibrium models we have reviewed above have the disadvantage that they do not automatically fit today's term structure of interest rates. The drift of the short rate as shown above is not usually a function of time. This leads us to the one-factor time-varying/arbitrage-free models. The essential difference between these two types of models is that in a time-varying/arbitrage-free model, the drift is, in general, dependent on time. This is because the shape of the initial zero curve governs the average path taken by the short rate in the future in an arbitrage-free model. In turns out that some equilibrium models can be converted to arbitrage-free models by including a function of time in the drift of the short rate.

### 3.2. One-factor Time-varying/Arbitrage-free Models

#### 3.2.1. Ho and Lee (1986)

Ho and Lee proposed the first arbitrage-free model of the term structure in a paper in 1986. They presented the model in the form of binomial tree of bond prices with two parameters: the

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short rate standard deviation and the market price of risk of the short rate. It has been shown by Jamshidian (1991a) that the continuous-time limit of the short rate is driven under the risk-neutral probability by the SDE:

$$dr(t) = \theta(t)dt + \sigma dW(t) \tag{9}$$

where the volatility term $\sigma$ is a constant and the drift term $\theta(t)$ is a function of time chosen to ensure the model fits the initial term structure.

#### 3.2.2. Hull and White One-factor Model (1990)

Hull and White explored an extension of the Vasicek (1977) model that provide an exact fit to the initial term structure.

$$dr(t) = \kappa\left(\frac{\theta(t)}{\kappa} - r(t)\right)dt + \sigma dW(t) \tag{10}$$

where $\kappa$ and $\sigma$ are constants. It can be characterized as the Ho-Lee model with mean reversion at rate $\kappa$ and the Vasicek model with a time-dependent reversion level. At time t, the short rate reverts to $\frac{\theta(t)}{\kappa}$ at rate $\kappa$.

#### 3.2.3. Hull and White (1993)

In this paper, they formed a general specification of the short rate:

$$dr(t) = \left(\theta(t) - \kappa r(t)\right)dt + \sigma r^{\alpha}(t)dW(t) \tag{11}$$

where $\kappa$ and $\sigma$ are constants, $\theta(t)$ is time-varying. There are a few existing models that can be extended in this framework. These models are such as the extended Vasicek model with $\alpha = 0$, the extended Cox, Ingersoll and Ross model with $\alpha = 0.5$ and the extended Brennan and Schwartz (1977) and Courtadon (1982) model with $\alpha = 1$.

The models we have seen so far (except Cox, Ingersoll and Ross model which has a square root process) are modeled as Gaussian processes; the popularity of using the Gaussian process is due to its analytical tractability. However, this process implies that there is a positive probability of negative rates. This leads to our next subsection of the one-factor model with lognormal rates to avoid the negative rates.

### 3.3. One-factor Time-varying Lognormal Models

#### 3.3.1. Black, Derman and Toy (1987)

Black, Derman and Toy proposed a one factor binomial model whose continuous time version has the form of:

$$d\ln r(t) = \left(\theta(t) - \kappa \ln r(t)\right)dt + \sigma_r dW(t) \tag{12}$$

It assumes a lognormal process for the short rate, which precludes negative value.

They extended the model to allow for time dependent volatility in the 1990 paper.

$$d\ln r(t) = \left(\theta(t) - \kappa \ln r(t)\right)dt + \sigma_r(t)dW(t) \tag{13}$$

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The model does not have as much analytical tractability as the Gaussian process model. It is not possible to produce formulas for valuing bonds in terms of the short rate using the model.

#### 3.3.2. Black and Karasinski (1991)

Black and Karasinski proposed an extension of the Black, Derman and Toy (1987) model with a time-varying mean reversion rate $\kappa(t)$. The model has the form of:

$$d\ln r(t) = \left(\theta(t) - \kappa(t)\ln r(t)\right)dt + \sigma_r(t)dW(t) \tag{14}$$

Again, the model lacks analytical properties.

Modeling lognormally distributed rates is the simplest way to avoid negative rates, but no closed form solution to the zero-coupon bonds can be found for these models.

To close this subsection, we list some other one-factor models that are not as popular as the ones we have discussed above. These models are Dothan (1978), Brennan and Schwartz (1977, 1980), Courtadon (1982), Rendleman and Bartter (1980) and Cox, Ingersoll and Ross (1980).

## 4. Multi-factor Models

So far we have reviewed the single factor models where the short rate is the only explanatory variable. Most of these models are characterized by their analytical tractability. However, these models often fail to match observed prices. For an economic point of view, it seems unreasonable to assume that the entire term structure is governed only by the short rate. So using more than one explanatory factor to model the interest rate is quite useful. Most multi-factor models are in fact based on two factors. These models are such as Cox, Ingersoll and Ross (1985b), Longstaff and Schwartz (1991), Fong and Vasicek (1991), Chen (1994) and Duffie and Kan (1996). In this subsection, we exam some popular multi-factor models in more detail.

### 4.1. Cox, Ingersoll and Ross (1985b) –CIR Model

Cox, Ingersoll and Ross presented a model in which the term structure of interest rates is determined by two factors: the real short rate $q(t)$ and the expected instantaneous inflation rate $\pi(t)$. Both factors are assumed to follow independent diffusion process:

$$\begin{aligned} dq(t) &= \mu_q(t)dt + \sigma_q(t)dW_q(t) \\ d\pi(t) &= \mu_\pi(t)dt + \sigma_\pi(t)dW_\pi(t) \end{aligned} \tag{15}$$

where $W_q$ and $W_q$ are two independent Brownian motions. They obtain a complicated, but analytical solution for the zero-coupon bond price. Similar framework is proposed by Brennan and Schwartz (1982), in which the term structure of interest rates depends on both the short term rate $r(t)$ and the long term rate $l(t)$.

### 4.2. Longstaff and Schwartz (1992)

Longstaff and Schwartz developed an equilibrium model of the economy and derived from there a two-factor term structure model. The two factors are the short term rate $r(t)$ and the variance of changes in the short term rate $v(t)$.

In their framework, the representative investor has a logarithmic utility and has the choice between investing and consuming the only good available in the economy, whose price $P(t)$ follows the SDE:

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$$d\frac{P(t)}{P(t)} = \left(\mu X(t) + \theta Y(t)\right)dt + \sigma\sqrt{Y}dW_1(t) \tag{16}$$

where $X(t)$ and $Y(t)$ are two specific economic factors. $X(t)$ is the expected return part that is unrelated to the Brownian motion $W_1(t)$; $Y(t)$ is the factor correlated with $dP(t)$. The dynamics of the two factors are given by:

$$\begin{aligned} dX(t) &= \left(a - bX(t)\right)dt + c\sqrt{X(t)}dW_2(t) \\ dY(t) &= \left(d - eY(t)\right)dt + f\sqrt{Y(t)}dW_3(t) \end{aligned} \tag{17}$$

where $W_2(t)$ and $W_3(t)$ are uncorrelated Brownian motions and $a,b,c,d,e,f > 0$.

Longstaff and Schwartz do not provide any intuitive interpretation for these two factors, but they show that $X(t)$ and $Y(t)$ can be related to observable quantities, as the equilibrium instantaneous interest rate $r(t)$ and the variance of its changes $v(t)$ are given by a weighted sum of these two factors.

$$\begin{aligned} r(t) &= \mu c^2 X(t) + \left(\theta - \sigma^2\right) f^2 Y(t) \\ v(t) &= \mu^2 c^4 X(t) + \left(\theta - \sigma^2\right)^2 f^4 Y(t) \end{aligned} \tag{18}$$

so that $r(t)$ and $v(t)$ are non-negative for all feasible values of state variables.

This model can be seen as an affine two factor model, in which one is the short rate and the other one is its volatility.

In a series of papers, Fong and Vasicek (1991, 1992a, 1992b) have derived a two-factor model using the same factors as this model. In their framework, the short rate and its variance evolve under the risk-neutral probability:

$$\begin{aligned} dr(t) &= \beta\left(\bar{r} - r(t)\right)dt + \sqrt{v(t)}dW_1(t) \\ dv(t) &= \varsigma\left(\bar{v} - v(t)\right)dt + \xi\sqrt{v(t)}dW_2(t) \end{aligned} \tag{19}$$

where $\bar{r}$ is the long term mean of the short rate and $v(t)$ is its instantaneous volatility, and $\bar{v}$ is the long term mean of the volatility. The two Brownian motions are correlated. This model does not preclude the positive probability of negative short term rates.

### 4.3. Chen (1996)

Chen (1996) proposed a three factor model of the term structure. In his model, the short rate dynamics depends on the current short rate, the stochastic mean of the short rate, and the stochastic volatility of the short rate.

$$\begin{aligned} dr(t) &= \kappa\left(\theta(t) - r(t)\right)dt + \sqrt{\sigma(t)}\sqrt{r(t)}dW_2(t) \\ d\theta &= v\left(\bar{\theta} - \theta(t)\right)dt + \xi\sqrt{\theta}dW_1(t) \\ d\sigma(t) &= \mu\left(\bar{\sigma} - \sigma(t)\right)dt + \eta\sqrt{\sigma}dW_3(t) \end{aligned} \tag{20}$$

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### 4.4. Duffie and Kan (1996)

Duffie and Kan introduced an N-factor affine term structure model which is obtained under the assumptions that the instantaneous short rate $r(t)$ is an affine function of a vector of unobserved state variables $\mathrm{X}(t)$.

$$r(t) = \rho_0 + \sum_{i=1}^{n} \rho_i \mathrm{X}_i(t) \equiv \rho_0 + \rho_X{}'\mathrm{X}(t) \tag{21}$$

The state variable vector follows an affine diffusion[41]:

$$d\mathrm{X}(t) = \kappa\left(\theta - \mathrm{X}(t)\right)dt + \Sigma\sqrt{S(t)}dW(t) \tag{22}$$

where W(t) is an N-dimensional independent standard Brownian motion under the risk-neutral probability, $\kappa$ and $\Sigma$ are N*N matrices, which may be non-diagonal and asymmetric, and S(t) is a diagonal matrix with the $i^{th}$ diagonal element given by:

$$\left[S(t)\right]_{ii} = \alpha_i + \beta_i \mathrm{X}(t) \tag{23}$$

Both the drifts in equation (22) and the conditional variances in equation (23) of the state variables are affine in X(t).

## 5. The Heath-Jarrow-Morton Model –HJM Model (1992)

Even with a multi-factor model, the term structure of interest rates has a rather limited number of degrees of freedom. An alternative approach to single and multi-factor interest rate modeling is to specify the entire term structure of interest rates. Rather than using a finite number of state variables, some authors use one state variable of infinite dimension, namely, the term structure itself. The first contribution to this approach was made by the Ho and Lee (1986) binomial model in a discrete time. It was the first to model movements in the entire term structure. Heath, Jarrow and Morton (1992) have significantly extended the Ho and Lee (1986) model by considering forward rates rather than bond prices as their building block; it also extended it from one factor model to a multi-factor model.

Heath, Jarrow and Morton framework firstly developed a class of models that are derived by directly modeling the dynamics of instantaneous forward rates. It models the entire term structure as a state variable, providing conditions in a general framework that incorporates all the principles of arbitrage-free pricing and zero-coupon bond dynamics. The Heath, Jarrow and Morton model shows that there is a link between the drift and standard deviation of an instantaneous forward rate. The drift of the forward rates under the risk-neutral probability is entirely determined by its volatility, which is the major contribution of this model.

$$df(t,T) = \sum_{i=1}^{n} \sigma_T(t,T)\int_t^T \sigma(t,\tau)d\tau + \sum_{i=1}^{n} \sigma_T(t,T)dW(t) \tag{24}$$

Since the short term rate is a specific forward rate, the short rate in the Heath, Jarrow and Morton model can be written in an integral form as:

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$$dr(t) = f(0,t) + \int_0^t \sigma_f(s,t) \int_0^t \sigma_f(s,u) duds + \int_0^t \sigma_f(s,t) dW(s) \tag{25}$$

Note that the difficulty of estimating the Heath, Jarrow and Morton model will arise because of the non-Markovin term in equation (25), which depends on the history of the process from time 0 to time t.

## 6. Advanced Term Structure Models

In this section, we highlight the most popular term structure models of bond yields that can be used as a foundation for further empirical research in the financial literature. The three main classes of term structure models are the affine term structure model; the Nelson-Siegel model and the quadratic term structure models: the affine term structure model, originally introduced by Duffie and Kan (1996), classified by Dai and Singleton[42] (2000) and extended to the essentially affine specification by Duffee (2002); the dynamic Nelson-Siegel model introduced by Diebold and Li (2006), which build on Nelson and Siegel (1987); and the class of quadratic term structure models classified by Ahn et al.(2002) and Leippold and Wu (2002).

The Nelson- Siegel model provides an intuitive description of the yield curve at each point in time. The dynamic Nelson-Siegel model (Diebold and Li, 2006) is easy to estimate and fits yield curve data well in-sample and produces good out-of sample forecasts. In contrast to arbitrage-free term structure models, this model class does not preclude arbitrage opportunities. However, an extension of the Nelson-Siegel model that is arbitrage-free does exist and this is done by Christensen, Diebold and Rudebusch (2007, 2008).

The affine and quadratic term structure models both have the arbitrage-free property; they derive the dynamic yield curve under a risk-neutral probability measure. The existence of risk-neutral probability measure implies that bond prices are arbitrage-free and the observed yield curve evolution is a result of the yield behavior under a historical probability measure. The transition from the risk-neutral to the historical measure is established via the market price of risk. Dai and Singleton (2000) provide the admissibility conditions and suggest a specification for completely affine term structure model. Duffee (2002) points out the restriction of the completely affine specification, and presents a broader class of essentially affine models, in which the market price of risk specification is more flexibly formulated.

Ahn et al. (2002) describe the classification and canonical representation of the quadratic term structure models analogously to the classification of affine models in Dai and Singleton (2000). They show that the quadratic model specification can capture the conditional volatility of yields better than the affine class.

Nyholm and Vidova-Koleva (2011) conduct an extensive out-of-sample forecasting experiment among quadratic, affine and dynamic Nelson-Siegel models using US yields curve monthly data from 1970 to 2000. They found that the quadratic three factor models provide the best in-sample-fit; the families of affine three-factor models and dynamic Nelson-Siegel models perform equally well in the out-of sample forecasting experiment, and that these two models produce better forecast than the quadratic model.

## 7. Conclusion

In this paper, we have reviewed a number of specifications of diffusion based term structure of interest rates models. (A summary of these models is provided in the Appendix). We have presented an overview of the most popular models by means of some general characteristics. From single factor to multi-factor models, forward rate based models and the most recent empirically advocated models; each of these models has its own advantages as well as

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disadvantages. On the whole, an ideal interest rate model should be theoretically consistent, flexible, well-specified and realistic; it should also provide good in-sample fit (to the data) and out-of-sample forecasting.

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[40] A principal component analysis is a statistical technique that identifies the best factors from historical yields, where the term best is in the sense of the two conditions: 1) the factors ought to explain a very large proportion of the variation of the yields of bonds at various horizons; 2) the factors should be independent of each other.

[41] Gaussian process and square-root process are the best known examples of affine diffusions. Gaussian process has a constant volatility, while the square-root processes introduce conditional heteroskedasticity by allowing the volatility function to depend on the state variables.

[42] Within the family of Duffie and Kan affine term structure model, there is a trade-off between flexibility in modeling the conditional correlations and volatilities of the risk factors. This trade-off is formalized by their classification of N-factor affine family into N + 1 non-nested subfamilies of models. Vasicek (1977), Chen (1996), and Cox, Ingersoll and Ross(1985) models are classified into distinct subfamilies of the affine models.

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# Appendix

**Table 1.** A partial list of the term structure models

| Model | Model Description |
|---|---|
| Merton (1970) | $dr(t) = \mu dt + \sigma dW(t)$ |
| Vasicek (1977) | $dr(t) = \kappa\left(\theta - r(t)\right)dt + \sigma dW(t)$ |
| Dothan (1978) | $dr(t) = \sigma r(t) dW(t)$ |
| Brennan and Schwartz (1979) | $dr(t) = \theta_1\left(r,\lambda,t\right)dt + \sigma_1\left(r,\lambda,t\right)dW_1(t)$ <br> $d\lambda(t) = \theta_2\left(r,\lambda,t\right)dt + \sigma_2\left(r,\lambda,t\right)dW_2(t)$ |
| Courtadon (1982) | $dr(t) = \kappa\left(\theta - r(t)\right)dt + \sigma r(t) dW(t)$ |
| Cox, Ingersoll and Ross (1985) | $dr(t) = \kappa\left(\theta - r(t)\right)dt + \sigma\sqrt{r(t)}dW(t)$ |
| Cox, Ingersoll and Ross (1985b) | $dq(t) = \mu_q(t)dt + \sigma_q(t)dW_q(t)$ <br> $d\pi(t) = \mu_\pi(t)dt + \sigma_\pi(t)dW_\pi(t)$ |
| Ho and Lee (1986) | $dr(t) = \theta(t)dt + \sigma dW(t)$ |
| Nelson and Siegel (1986) | $f(\tau) = \beta_1 + \beta_2 e^{-\lambda\tau} + \beta_3 \frac{1}{\lambda\tau} e^{-\lambda\tau}$ |
| Hull and White (1990) | $dr(t) = \kappa\left(\frac{\theta(t)}{\kappa} - r(t)\right)dt + \sigma dW(t)$ |
| Black and Karasinski (1992) | $d\ln r(t) = \left(\theta(t) - \kappa(t)\ln r(t)\right)dt + \sigma_r(t)dW(t)$ |
| Hull and White (1993) | $dr(t) = \left(\theta(t) - \kappa r(t)\right)dt + \sigma r^{\alpha}(t)dW(t)$ |
| Black, Derman and Toy (1990) | $d\ln r(t) = \left(\theta(t) - \kappa\ln r(t)\right)dt + \sigma_r dW(t)$ |
| Heath, Jarrow and Morton (1992) | $df(t,T) = \sum_{i=1}^{n}\sigma_T(t,T)\int_t^T \sigma(t,\tau)d\tau + \sum_{i=1}^{n}\sigma_T(t,T)dW(t)$ |
| Longstaff and Schwartz (1992) | $dX(t) = \left(a - bX(t)\right)dt + c\sqrt{X(t)}dW_1(t)$ <br> $dY(t) = \left(d - eY(t)\right)dt + f\sqrt{Y(t)}dW_2(t)$ |
| and Chen and Scott (1992) | |
| Fong and Vasicek (1991, 1992a, 1992b) | $dr(t) = \beta\left(\bar{r} - r(t)\right)dt + \sqrt{v(t)}dW_1(t)$ <br> $dv(t) = \varsigma\left(\bar{v} - v(t)\right)dt + \xi\sqrt{v(t)}dW_2(t)$ |
| Chen (1996) | $dr(t) = \kappa\left(\theta(t) - r(t)\right)dt + \sqrt{\sigma(t)}\sqrt{r(t)}dW_2(t)$ <br> $d\theta = v\left(\bar{\theta} - \theta(t)\right)dt + \xi\sqrt{\theta}dW_1(t)$ <br> $d\sigma(t) = \mu\left(\bar{\sigma} - \sigma(t)\right)dt + \eta\sqrt{\sigma}dW_3(t)$ |
| Duffie and Kan (1996) | $d\mathrm{X}(t) = \kappa(\theta - \mathrm{X}(t)dt + \Sigma\sqrt{S(t)}dW(t)$ <br> $r(t) = \rho_0 + \rho_X{}'\mathrm{X}(t)$ |
| Ahn et al. (2002) | $d\mathrm{X}(t) = \kappa(\theta - \mathrm{X}(t)dt + \Sigma dW(t)$ <br> $r(t) = \alpha + \beta'\mathrm{X}(t) + \mathrm{X}(t)'\psi\,\mathrm{X}(t)$ |
| Christensen et al. (2007) | $d\mathrm{X}(t) = \kappa(\theta - \mathrm{X}(t)dt + \Sigma dW(t)$ <br> $r(t) = L(t) + S(t)$ |