

Colored Correlated Noises Induced Regime Shifts in a Time-delayed Lake Eutrophication Ecosystem

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Abstract

The catastrophic regime shifts in the lake eutrophication ecosystems may be usually caused by environmental perturbations, and the lake dynamics exhibit bistability characteristics with oligotrophic and eutrophic states. This paper is devoted to investigate the time delay effect in a time-delayed lake eutrophication ecosystem (TD-LEE), where the dynamics ecosystem is deemed to be disturbed by both multiplicative and additive noises. On the basis of small delay approximation, we derive the expression of the generalized effective potential function in the TD-LEE and discuss the impact of stochastic delay on the regime shifts between oligotrophic and eutrophic states. In the view of dynamical behaviors on the TD-LEE model, we also revealed that the combination with time delay and colored correlated noises can expedite the regime shift from the eutrophication to oligotrophication, thereby inhibiting the development of lake eutrophication and protecting the stability of lake ecosystem.

Keywords

Colored Correlated Noises; Time-delayed Lake Eutrophication Ecosystem; Phosphorus Cycle; Generalized Effective Potential Function.

1. Introduction

The ecological problems of lake eutrophication require the solution of the so-called dynamics equation of phosphorus cycle. It is a nonlinear partial differential equation, which is difficult to obtain its analytical solution. The lake eutrophication ecosystems usually exhibit bistable states with oligotrophic and eutrophic states, while the catastrophic regime shifts due to the lake bistability may be caused by the environmental noises or perturbations. It is well known that the bistability induced rich dynamics has been widely studied in the ecological systems, for example, bistability with eutrophic and oligotrophic states, bistability with vegetational and desert states, bistability with extinct and tumorigenic states, and so on [1-5]. Ecosystems sometimes suddenly shift from one steady state to another, this process may lead to degradation of ecosystem services and consequent economic losses. Generally, the regime shifts in ecological systems are irreversible, which inspires us to explore the dynamic ecological mechanism to prevent irreversible ecological damage.

Many existing ecological dynamic analysis approaches has focused on the deterministic ecological systems, which is easy to obtain analytical solutions in mathematical form [6-8]. In real ecosystems, it is filled with intrinsic stochasticity related to the ceaselessly changing and interfering ecology participants [9, 10]. However, the gradual external factors rather than huge exogenous shocks induced the sudden steady-state transition, this is called catastrophic regime shifts. In recent years, environmental noises has received great attention, especially in

dynamics and resilience of ecological systems [11-13]. For example, Wang et al. explored the mean-field vegetation model subject to environmental noises, where the interaction between noises induced the resonant phenomenon [11]. Subsequently, Zeng et al. investigated the multiplicative and additive noises in lake approaching eutrophication ecosystems, it is found that the hysteresis loop acreage can be broadened by the correlated noises [12]. In addition, under large external impacts, Wang et al. investigated the phosphorus dynamics behavior and elaborated the flickering gives early warning information before the catastrophic eutrophic transition [13].

On the other hand, it is worth noting that the dynamic behaviors of nonlinear systems with time-delay and various noises have been recently investigated, which revealed significant results that the time delay can induce rich dynamic behaviors [14-17]. In the nature, the transmission of information, matter, energy of the nonlinear systems inevitably takes certain time, so it is more reasonable to investigate the time-delayed effect in some actual systems. To address this problem, Zeng et al. has explored the influence of intrinsic and extrinsic noises on the dynamic behavior to take advantage of the time-delayed vegetation model and the time-delayed tumor cells model, the results show that time delay and noises could induce resonance phenomenon in the nonlinear systems driven by a weak multiplicative periodic force [18-20]. More interestingly, Zeng et al. demonstrated the input and loss noises can be considered as the controllable parameters on regime shifts of oligotrophication and eutrophication in a time-delayed lake ecosystem [21]. Jia and Mei have also studied a delay-induced catastrophic regime shifts in a metastable system with two different kinds of time delays and cross-correlated noises [22]. Moreover, Wang et al. have also investigated stochastic resonance in paced time-delayed neuronal networks and presented non-trivial effects induced by finite delays [23-25]. However, these research findings are only investigated in nonlinear systems with delay and noise, while the multiple noises effect with colored correlation has almost not been considered. In real ecology, the effect of the environmental noises with colored correlation should be taken into account when designing the optimal lake environment monitoring. Besides, the synergistic effect between colored correlated noises and time delay on lake ecosystems could be meaningful, since the lake environment protection strategy is updated only when the colored condition is considered. This paper aims to explore the dynamic behavior in a time-delayed lake eutrophication ecosystem (TD-LEE) model, where the multiplicative noise and additive noise are considered to be colored correlation. In comparison with the deterministic scenario, the hysteresis loop acreage in the TD-LEE model can be broaden by the colored correlated noises, which is consistent with the phenomenon shown in [2] and [21]. Later, on the basis of the small delay approximation, the expression of the generalized effective potential function is deduced. Meanwhile, the impact of colored correlated noise and stochastic delay on the regime shifts between oligotrophic and eutrophic states is discussed.

2. The Colored Correlated Noises Induced Regime Shifts in the TD-LEE Model

The lake eutrophication model is usually characterized by phosphorus cycling, the watershed and recyclable in sediment provide the phosphorus intake and the deposition, efflux and sequestration of the consumer or benthic biomass offer phosphorus output. In accordance with the Carpenter's model of lake dynamics [6], we can obtain the deterministic dynamical equation of the lake eutrophication model as follows:

$$\frac{dp}{dt} = c - mp + rs(p) \quad (1)$$

where the term c denotes the phosphorus intake rate, p denotes phosphorus concentration in the lake water, m denotes the phosphorus loss rate. The term r denotes the recycling coefficient in concern with hypolimnion being anoxic, and it can be explained as the maximum rate of recyclable phosphorus. The so-called Hill function term $s(p)$ that induces two alternative regimes is used to represent ecological bistability, which is consist of a high phosphorus concentration that recognizes as the eutrophic state and a low phosphorus concentration that equals in the oligotrophic state.

In the first instance, we introduce the dynamics behavior of the phosphorus concentration in the deterministic lake ecosystem. In the case of $r = 0$, only one equilibrium point (c/m) can be achieved. It is noted that the Hill term $s(p)$ can induce two alternative stable states, which is increased rapidly on the threshold w :

$$s(p) = \frac{p^w}{z^w + p^w} \quad (2)$$

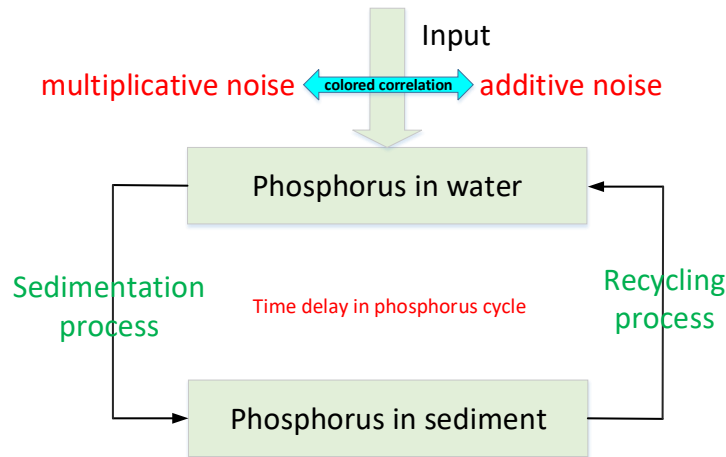


Figure 1. Phosphorus flows in the TD-LEE model.

Due to the influence of environmental perturbations on phosphorus intake such as dry deposition, rainfall and groundwater, multiplicative and additive noises with colored correlation inevitably participate in the phosphorus dynamics. For simplicity, we chose to introduce the phosphorus flows in the TD-LEE model in Figure 1. Suppose the initial parameters $m = 1$, $z = 1$ and $w = 8$, the Langevin equation corresponding to Eq. (1) can be rewritten as:

$$\frac{dp(t)}{dt} = c - p(t - \beta) + \frac{rp^8}{1 + p^8} + p\xi(t) + \eta(t) \quad (3)$$

where $p(t - \beta)$ denotes the time-delayed variable with $p_\beta(t) = p(t - \beta)$, $\xi(t)$ and $\eta(t)$ are the multiplicative Gaussian noise and additive Gaussian noise with colored correlation.

$$\langle \xi(t) \rangle = \langle \eta(t) \rangle = 0 \quad (4)$$

$$\langle \xi(t) \xi(t') \rangle = 2D\delta(t - t') \quad (5)$$

$$\langle \eta(t) \eta(t') \rangle = 2Q\delta(t-t') \quad (6)$$

$$\langle \xi(t) \eta(t') \rangle = \langle \eta(t) \xi(t') \rangle = \frac{\lambda\sqrt{DQ}}{\tau} \exp\left(-\frac{|t-t'|}{\tau}\right) \quad (7)$$

where D and Q are the intensities of the multiplicative Gaussian noise and additive Gaussian noise respectively. The parameters λ and τ denote the cross-correlation time and intensity of the colored correlated noises.

3. The Generalized Effective Potential Function for the TD-LEE Model

Based on the small delay approximation, the one-dimensional non-Markov process can be converted into the corresponding Markov one [12]. Then, the TD-LEE model Eq. (3) can be rewritten as:

$$\frac{dp(t)}{dt} = f_{\text{eff}}(p) + g_{\text{eff}}(p)\xi(t) + \eta(t) \quad (8)$$

It is noting that the subscript eff stands for the “effective”, and the effective coefficient $f_{\text{eff}}(p)$ and $g_{\text{eff}}(p)$ can be expressed as:

$$f_{\text{eff}}(p) = \int_{-\infty}^{+\infty} \left(c - p + \frac{rp^8}{1+p^8} \right) P_1(p_\beta, t - \beta | p, t) dp_\beta \quad (9)$$

$$g_{\text{eff}}(p) = \int_{-\infty}^{+\infty} p P_2(p_\beta, t - \beta | p, t) dp_\beta \quad (10)$$

where $P_1(p_\beta, t - \beta | p, t)$ and $P_2(p_\beta, t - \beta | p, t)$ represents the conditional probability density of the stochastic process $p(t)$.

$$P_1(p_\beta, t - \beta | p, t) = \sqrt{\frac{1}{2\pi G(p)\beta}} \exp\left\{-\frac{[p_\beta - p - f(p)\beta]^2}{2G(p)\beta}\right\} \quad (11)$$

$$P_2(p_\beta, t - \beta | p, t) = \sqrt{\frac{1}{2\pi G(p)\beta}} \exp\left\{-\frac{[p_\beta - p - g(p)\beta]^2}{2G(p)\beta}\right\} \quad (12)$$

With:

$$G(p) = Dp^2 + \frac{2\lambda}{1+2\tau}\sqrt{DQ}p + Q \quad (13)$$

$$f(p) = c - p + \frac{rp^8}{1+p^8} \quad (14)$$

$$g(p) = p \quad (15)$$

Inserting Eqs. (11) and (12) into (9) and (10), we can obtain.

$$f_{\text{eff}}(p) = \left(c - p + \frac{rp^8}{1+p^8} \right) (1+\beta) \quad (16)$$

$$g_{\text{eff}}(p) = p(1+\beta) \quad (17)$$

In accordance with [11], the approximate delay Fokker-Planck equation of Eq. (3) is further reduced to:

$$\frac{\partial P_{st}(p,t)}{\partial t} = -\frac{\partial}{\partial p} \mu(p) P_{st}(p,t) + \frac{\partial^2}{\partial p^2} \sigma^2(p) P_{st}(p,t) \quad (18)$$

Where:

$$\mu(p) = \left(f(p) + Dp + \frac{\lambda}{1+2\tau} \sqrt{DQ} \right) (1+\beta) \quad (19)$$

$$\sigma^2(p) = \left(Dp^2 + \frac{2\lambda}{1+2\tau} \sqrt{DQ} p + Q \right) (1+\beta)^2 \quad (20)$$

The steady probability distribution of the Fokker-Planck equation of the TD-LEE model is given by:

$$\begin{aligned} P_{st}(p) &= \frac{N}{\sigma^2(p)} \exp \left\{ \int^p \frac{\mu(p)}{\sigma^2(p)} dp \right\} \\ &= \frac{N}{\sigma(p)} \exp \left\{ \int^p \frac{f_{\text{eff}}(p)}{\sigma^2(p)} dp \right\} \\ &= \frac{N}{\sigma(p)} \exp \left\{ -\frac{U_\beta(p)}{D} \right\} \end{aligned} \quad (21)$$

where N is the normalization constant, and $U_\beta(p)$ is the generalized effective potential function with the form as follows:

$$\begin{aligned}
U_{\beta}(p) = A_0 \times & \left\{ \frac{8 \arctan[(K + Dp)/\sqrt{-K^2 + DQ}]}{D\sqrt{-K^2 + DQ}} \times A_1 + A_2 \right. \\
& + 2r \arctan\left(p \csc\left[\frac{\pi}{8}\right] + \cot\left[\frac{\pi}{8}\right]\right) \times A_3 + 2r \arctan\left(p \csc\left[\frac{\pi}{8}\right] - \cot\left[\frac{\pi}{8}\right]\right) \times A_4 \\
& + 2r \arctan\left(p \sec\left[\frac{\pi}{8}\right] + \tan\left[\frac{\pi}{8}\right]\right) \times A_5 + 2r \arctan\left(p \sec\left[\frac{\pi}{8}\right] - \tan\left[\frac{\pi}{8}\right]\right) \times A_6 \\
& - r \ln\left(1 + p^2 + 2p \cos\left[\frac{\pi}{8}\right]\right) \times A_7 - r \ln\left(1 + p^2 - 2p \cos\left[\frac{\pi}{8}\right]\right) \times A_8 \\
& \left. - r \ln\left(1 + p^2 + 2p \sin\left[\frac{\pi}{8}\right]\right) \times A_9 + r \ln\left(1 + p^2 - 2p \sin\left[\frac{\pi}{8}\right]\right) \times A_{10} \right\} \quad (22)
\end{aligned}$$

in which $K = \lambda\sqrt{DQ}/(1+2\tau)$ and the terms $A_0 - A_{10}$ are given as follows:

$$A_0 = -D/\left[8(1+\beta)(D^8 + 256K^8 - 512DK^6Q + 320D^2K^4Q^2 - 64D^3K^2Q^3 + 2D^4Q^4 + Q^8)\right]$$

$$\begin{aligned}
A_1 = & D^8K + K(256K^8 + Q^8) \\
& + cD(D^8 + 256K^8 - 512DK^6Q + 320D^2K^4Q^2 - 64D^3K^2Q^3 + 2D^4Q^4 + Q^8) \\
& + D^5Q^4r + 2D^4KQ^3(Q - 16Kr) - 64D^2K^5Q(-5Q + 4Kr) \\
& + 32D^3K^3Q^2(-2Q + 5Kr) + D(-512K^7Q + 128K^8r + Q^8r)
\end{aligned}$$

$$\begin{aligned}
A_2 = & 8Kr \ln(1 + p^8) \times (16K^6 - 24DK^4Q + 10D^2K^2Q^2 - D^3Q^3) \\
& - 4 \ln(Dp^2 + 2Kp + Q) \times [D^7 + 256K^8/D + Q^8/D + 2D^3Q^3(Q - 4Kr) \\
& + 128K^6(-4Q + Kr) - 64DK^4Q(-5Q + 3Kr) + 16D^2K^2Q^2(-4Q + 5Kr)]
\end{aligned}$$

$$\begin{aligned}
A_3 = & D^6(\sqrt{2}K - Q \cos[\frac{\pi}{8}]) + D^5(-4KQ + 4K^2 \cos[\frac{\pi}{8}] + Q^2 \cos[\frac{\pi}{8}]) + D^7 \sin[\frac{\pi}{8}] \\
& + D(16\sqrt{2}K^3Q^3 + 4KQ^5 + 12K^2Q^4 \cos[\frac{\pi}{8}] + Q^6 \cos[\frac{\pi}{8}] + 64K^6 \sin[\frac{\pi}{8}] + 80K^4Q^2 \sin[\frac{\pi}{8}]) \\
& + D^4(8K^3 + 3\sqrt{2}KQ^2 - 12K^2Q \cos[\frac{\pi}{8}] - Q^3 \sin[\frac{\pi}{8}]) \\
& + D^2(16\sqrt{2}K^5 - 3\sqrt{2}KQ^4 - Q^5 \cos[\frac{\pi}{8}] - 80K^4Q \sin[\frac{\pi}{8}] - 24K^2Q^3 \sin[\frac{\pi}{8}]) \\
& + D^3(-16\sqrt{2}K^3Q + 16K^4 \cos[\frac{\pi}{8}] + 24K^2Q^2 \sin[\frac{\pi}{8}] + Q^4 \sin[\frac{\pi}{8}]) \\
& - Q(16\sqrt{2}K^5Q + 8K^3Q^3 + \sqrt{2}KQ^5 + 16K^4Q^2 \cos[\frac{\pi}{8}] + 4K^2Q^4 \cos[\frac{\pi}{8}] + 64K^6 \sin[\frac{\pi}{8}] + Q^6 \sin[\frac{\pi}{8}])
\end{aligned}$$

$$\begin{aligned}
A_4 = & -D^6(\sqrt{2}K + Q \cos[\frac{\pi}{8}]) + D^5(4KQ + 4K^2 \cos[\frac{\pi}{8}] + Q^2 \cos[\frac{\pi}{8}]) + D^7 \sin[\frac{\pi}{8}] \\
& + D(-16\sqrt{2}K^3Q^3 - 4KQ^5 + 12K^2Q^4 \cos[\frac{\pi}{8}] + Q^6 \cos[\frac{\pi}{8}] + 64K^6 \sin[\frac{\pi}{8}] + 80K^4Q^2 \sin[\frac{\pi}{8}]) \\
& - D^4(8K^3 + 3\sqrt{2}KQ^2 + 12K^2Q \cos[\frac{\pi}{8}] + Q^3 \sin[\frac{\pi}{8}]) \\
& - D^2(16\sqrt{2}K^5 - 3\sqrt{2}KQ^4 + Q^5 \cos[\frac{\pi}{8}] + 80K^4Q \sin[\frac{\pi}{8}] + 24K^2Q^3 \sin[\frac{\pi}{8}]) \\
& + D^3(16\sqrt{2}K^3Q + 16K^4 \cos[\frac{\pi}{8}] + 24K^2Q^2 \sin[\frac{\pi}{8}] + Q^4 \sin[\frac{\pi}{8}]) \\
& + Q(16\sqrt{2}K^5Q + 8K^3Q^3 + \sqrt{2}KQ^5 - 16K^4Q^2 \cos[\frac{\pi}{8}] - 4K^2Q^4 \cos[\frac{\pi}{8}] - 64K^6 \sin[\frac{\pi}{8}] - Q^6 \sin[\frac{\pi}{8}])
\end{aligned}$$

$$\begin{aligned}
A_5 = & D^7 \cos[\frac{\pi}{8}] + D^3(-16\sqrt{2}K^3Q + 24K^2Q^2 \cos[\frac{\pi}{8}] + Q^4 \cos[\frac{\pi}{8}] - 16K^4 \sin[\frac{\pi}{8}]) \\
& + D^6(\sqrt{2}K + Q \sin[\frac{\pi}{8}]) + D^4(-8K^3 + 3\sqrt{2}KQ^2 - Q^3 \cos[\frac{\pi}{8}] + 12K^2Q \sin[\frac{\pi}{8}]) \\
& - D^5(-4KQ + 4K^2 \sin[\frac{\pi}{8}] + Q^2 \sin[\frac{\pi}{8}]) \\
& - Q(16\sqrt{2}K^5Q - 8K^3Q^3 + \sqrt{2}KQ^5 + 64K^6 \cos[\frac{\pi}{8}] + Q^6 \cos[\frac{\pi}{8}] - 16K^4Q^2 \sin[\frac{\pi}{8}] - 4K^2Q^4 \sin[\frac{\pi}{8}]) \\
& + D^2(16\sqrt{2}K^5 - 3\sqrt{2}KQ^4 - 80K^4Q \cos[\frac{\pi}{8}] - 24K^2Q^3 \cos[\frac{\pi}{8}] + Q^5 \sin[\frac{\pi}{8}]) \\
& + D(16\sqrt{2}K^3Q^3 - 4KQ^5 + 64K^6 \cos[\frac{\pi}{8}] + 80K^4Q^2 \cos[\frac{\pi}{8}] - 12K^2Q^4 \sin[\frac{\pi}{8}] - Q^6 \sin[\frac{\pi}{8}])
\end{aligned}$$

$$\begin{aligned}
A_6 = & D^7 \cos[\frac{\pi}{8}] + D^3(16\sqrt{2}K^3Q + 24K^2Q^2 \cos[\frac{\pi}{8}] + Q^4 \cos[\frac{\pi}{8}] - 16K^4 \sin[\frac{\pi}{8}]) \\
& + D^6(-\sqrt{2}K + Q \sin[\frac{\pi}{8}]) + D^4(8K^3 - 3\sqrt{2}KQ^2 - Q^3 \cos[\frac{\pi}{8}] + 12K^2Q \sin[\frac{\pi}{8}]) \\
& - D^5(4KQ + 4K^2 \sin[\frac{\pi}{8}] + Q^2 \sin[\frac{\pi}{8}]) \\
& + Q(16\sqrt{2}K^5Q - 8K^3Q^3 + \sqrt{2}KQ^5 - 64K^6 \cos[\frac{\pi}{8}] - Q^6 \cos[\frac{\pi}{8}] + 16K^4Q^2 \sin[\frac{\pi}{8}] + 4K^2Q^4 \sin[\frac{\pi}{8}]) \\
& + D^2(-16\sqrt{2}K^5 + 3\sqrt{2}KQ^4 - 80K^4Q \cos[\frac{\pi}{8}] - 24K^2Q^3 \cos[\frac{\pi}{8}] + Q^5 \sin[\frac{\pi}{8}]) \\
& + D(-16\sqrt{2}K^3Q^3 + 4KQ^5 + 64K^6 \cos[\frac{\pi}{8}] + 80K^4Q^2 \cos[\frac{\pi}{8}] - 12K^2Q^4 \sin[\frac{\pi}{8}] - Q^6 \sin[\frac{\pi}{8}])
\end{aligned}$$

$$\begin{aligned}
A_7 = & D^7 \cos\left[\frac{\pi}{8}\right] + D^5(4K^2 - Q^2) \sin\left[\frac{\pi}{8}\right] + D^4 Q(-3\sqrt{2}KQ + Q^2 \cos\left[\frac{\pi}{8}\right] + 12K^2 \sin\left[\frac{\pi}{8}\right]) \\
& + D^3(16\sqrt{2}K^3Q - 24K^2Q^2 \cos\left[\frac{\pi}{8}\right] + Q^4 \cos\left[\frac{\pi}{8}\right] - 16K^4 \sin\left[\frac{\pi}{8}\right]) + D^6(\sqrt{2}K - Q \sin\left[\frac{\pi}{8}\right]) \\
& + Q(-16\sqrt{2}K^5Q + \sqrt{2}KQ^5 - 64K^6 \cos\left[\frac{\pi}{8}\right] + Q^6 \cos\left[\frac{\pi}{8}\right] - 16K^4Q^2 \sin\left[\frac{\pi}{8}\right] + 4K^2Q^4 \sin\left[\frac{\pi}{8}\right]) \\
& - D^2(16\sqrt{2}K^5 + 3\sqrt{2}KQ^4 - 80K^4Q \cos\left[\frac{\pi}{8}\right] + 24K^2Q^3 \cos\left[\frac{\pi}{8}\right] + Q^5 \sin\left[\frac{\pi}{8}\right]) \\
& + D(16\sqrt{2}K^3Q^3 - 64K^6 \cos\left[\frac{\pi}{8}\right] + 80K^4Q^2 \cos\left[\frac{\pi}{8}\right] + 12K^2Q^4 \sin\left[\frac{\pi}{8}\right] - Q^6 \sin\left[\frac{\pi}{8}\right])
\end{aligned}$$

$$\begin{aligned}
A_8 = & -D^7 \cos\left[\frac{\pi}{8}\right] + D^5(-4K^2 + Q^2) \sin\left[\frac{\pi}{8}\right] - D^4 Q(3\sqrt{2}KQ + Q^2 \cos\left[\frac{\pi}{8}\right] + 12K^2 \sin\left[\frac{\pi}{8}\right]) \\
& + D^3(16\sqrt{2}K^3Q + 24K^2Q^2 \cos\left[\frac{\pi}{8}\right] - Q^4 \cos\left[\frac{\pi}{8}\right] + 16K^4 \sin\left[\frac{\pi}{8}\right]) + D^6(\sqrt{2}K + Q \sin\left[\frac{\pi}{8}\right]) \\
& + Q(-16\sqrt{2}K^5Q + \sqrt{2}KQ^5 + 64K^6 \cos\left[\frac{\pi}{8}\right] - Q^6 \cos\left[\frac{\pi}{8}\right] + 16K^4Q^2 \sin\left[\frac{\pi}{8}\right] - 4K^2Q^4 \sin\left[\frac{\pi}{8}\right]) \\
& + D^2(-16\sqrt{2}K^5 - 3\sqrt{2}KQ^4 - 80K^4Q \cos\left[\frac{\pi}{8}\right] + 24K^2Q^3 \cos\left[\frac{\pi}{8}\right] + Q^5 \sin\left[\frac{\pi}{8}\right]) \\
& + D(16\sqrt{2}K^3Q^3 + 64K^6 \cos\left[\frac{\pi}{8}\right] - 80K^4Q^2 \cos\left[\frac{\pi}{8}\right] - 12K^2Q^4 \sin\left[\frac{\pi}{8}\right] + Q^6 \sin\left[\frac{\pi}{8}\right])
\end{aligned}$$

$$\begin{aligned}
A_9 = & D^5(-4K^2 + Q^2) \cos\left[\frac{\pi}{8}\right] + D^6(-\sqrt{2}K + Q \cos\left[\frac{\pi}{8}\right]) + D^7 \sin\left[\frac{\pi}{8}\right] \\
& + D^4(3\sqrt{2}KQ^2 - 12K^2Q \cos\left[\frac{\pi}{8}\right] + Q^3 \sin\left[\frac{\pi}{8}\right]) \\
& + D(-16\sqrt{2}K^3Q^3 - 12K^2Q^4 \cos\left[\frac{\pi}{8}\right] + Q^6 \cos\left[\frac{\pi}{8}\right] - 64K^6 \sin\left[\frac{\pi}{8}\right] + 80K^4Q^2 \sin\left[\frac{\pi}{8}\right]) \\
& + D^2(16\sqrt{2}K^5 + 3\sqrt{2}KQ^4 + Q^5 \cos\left[\frac{\pi}{8}\right] + 80K^4Q \sin\left[\frac{\pi}{8}\right] - 24K^2Q^3 \sin\left[\frac{\pi}{8}\right]) \\
& + D^3(-16\sqrt{2}K^3Q + 16K^4 \cos\left[\frac{\pi}{8}\right] - 24K^2Q^2 \sin\left[\frac{\pi}{8}\right] + Q^4 \sin\left[\frac{\pi}{8}\right]) \\
& + Q(16\sqrt{2}K^5Q - \sqrt{2}KQ^5 + 16K^4Q^2 \cos\left[\frac{\pi}{8}\right] - 4K^2Q^4 \cos\left[\frac{\pi}{8}\right] - 64K^6 \sin\left[\frac{\pi}{8}\right] + Q^6 \sin\left[\frac{\pi}{8}\right])
\end{aligned}$$

$$\begin{aligned}
A_{10} = & D^5(-4K^2 + Q^2) \cos\left[\frac{\pi}{8}\right] + D^6(\sqrt{2}K + Q \cos\left[\frac{\pi}{8}\right]) + D^7 \sin\left[\frac{\pi}{8}\right] \\
& + D^4(-3\sqrt{2}KQ^2 - 12K^2Q \cos\left[\frac{\pi}{8}\right] + Q^3 \sin\left[\frac{\pi}{8}\right]) \\
& + D(16\sqrt{2}K^3Q^3 - 12K^2Q^4 \cos\left[\frac{\pi}{8}\right] + Q^6 \cos\left[\frac{\pi}{8}\right] - 64K^6 \sin\left[\frac{\pi}{8}\right] + 80K^4Q^2 \sin\left[\frac{\pi}{8}\right]) \\
& + D^2(-16\sqrt{2}K^5 - 3\sqrt{2}KQ^4 + Q^5 \cos\left[\frac{\pi}{8}\right] + 80K^4Q \sin\left[\frac{\pi}{8}\right] - 24K^2Q^3 \sin\left[\frac{\pi}{8}\right]) \\
& + D^3(16\sqrt{2}K^3Q + 16K^4 \cos\left[\frac{\pi}{8}\right] - 24K^2Q^2 \sin\left[\frac{\pi}{8}\right] + Q^4 \sin\left[\frac{\pi}{8}\right]) \\
& + Q(-16\sqrt{2}K^5Q + \sqrt{2}KQ^5 + 16K^4Q^2 \cos\left[\frac{\pi}{8}\right] - 4K^2Q^4 \cos\left[\frac{\pi}{8}\right] - 64K^6 \sin\left[\frac{\pi}{8}\right] + Q^6 \sin\left[\frac{\pi}{8}\right])
\end{aligned}$$

4. Dynamic Behaviors Over the TD-LEE Model with Colored Correlated Noises

Typically, in the previous reports of lake eutrophication, Zeng et al. explore the white cross-correlation relationship between the input and recycling perturbations with Gaussian white noises in the lake eutrophication system model and different time delay types induced regime shifts in lake [20, 21]. Inspired by these, this paper is devoted to investigate the effect of the colored correlated noises on the dynamic behaviors in the time-delayed lake eutrophication ecosystem model.

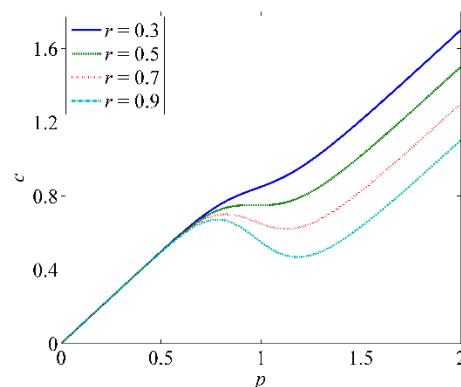


Figure 2. The rate of phosphorus intake c varies with the phosphorus concentration p under different recycling coefficient r in the deterministic lake model

Figure 2 shows that the S shape bifurcation diagram for the rate of phosphorus intake c as a function of phosphorus concentration p in the deterministic model, and the ecological bistability is exhibited by the various recycling parameter r . Following the idea in [20], the fixed parameters as $m = 1$, $z = 1$ and $w = 8$ is set to display the various region of ecological bistability. It can be seen that the ecological bistability vanishes when the recycling coefficient fall into relatively low case, i.e., $r < 0.5$. Moreover, the ecological bistability phenomenon can be awakened by the case of recycling coefficient exceeds 0.5. With the increasing of recycling coefficient, the region of ecological bistability can be enlarged. Thus, in the absence of environmental fluctuations, the recycling coefficient term is crucial for lake eutrophication ecosystem model to obtain the ecological bistability.

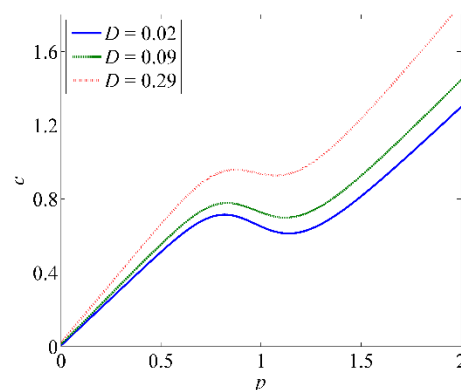


Figure 3. The rate of phosphorus intake c varies with the phosphorus concentration p under different multiplicative Gaussian noise intensity D in the time-delayed lake eutrophication ecosystem model, the other parameters are $Q = 0.08$, $\lambda = 0.2$, $\tau = 0.2$, $r = 0.75$.

Because the noises in lake can affect the rate of phosphorus intake in the watershed continually fluctuates, the multiplicative and additive noises with colored correlation should be employed to classify the S shape bifurcation in the time-delayed lake eutrophication ecosystem model. In Fig. 3, The S shape bifurcation diagram for the rate of phosphorus intake c as a function of the phosphorus concentration p under different multiplicative Gaussian noise intensity D in the TD-LEE model is shown to exhibit the colored correlated noises effect in inducing regime shifts between oligotrophic state and eutrophic state. It is demonstrated that the ecological bistability occurs with the colored correlated noises in the TD-LEE model. More interestingly, the hysteresis loop acreage will enlarge with the reduction of multiplicative noise intensity, and it would be alter the S bifurcation diagram. In addition, the colored correlated noises in the TD-LEE model would like to switch from the eutrophic state to the oligotrophic state, it rewards us to explore the dynamic behavior for providing a lake conservation strategy.

Through the above discussion of bifurcation diagrams, it is demonstrated that the colored correlated noises can cause the ecological bistability in the TD-LEE model due to the expanding hysteresis loop area. The colored correlated noises induced two distinguishing features of the oligotrophic state and the eutrophic state. The ecological bistability is a double-edged sword, where lake is at risk of shifting from oligotrophication to eutrophication and it is also acquires an opportunity to facilitate ecological restoration from eutrophic state under certain conditions. In consequence, it is dedicated to discuss how colored correlated noises induced regime shifts from the eutrophic state to the oligotrophic one.

In order to elaborate that the lake ecosystem maintaining in the desired oligotrophic state with the aid of input noise, we draw the graphic descriptions of generalized effective potential function in which the nutritional status of phosphorus can be switched using the controlling parameter of colored correlated noises in the TD-LEE model. Based on the ecological physics, the dynamical analyses of the phosphorus concentration in the lake eutrophication ecosystem are mostly dependent on the generalized force. Meanwhile, it can be seen that the generalized force is the negative derivative of the generalized effective potential in the TD-LEE model. The valley bottom of the left and right in the generalized effective potential represent the two stable states, while the wave crest one represents the unstable state. It is known that the two stable states exhibit the low nutrient phosphorus and high nutrient phosphorus and the unstable state exhibit the valley in phosphorus concentration. Therefore, we explore the possibility phenomenon that the colored correlated noises-induced regime shifts from the eutrophic state to the oligotrophic one.

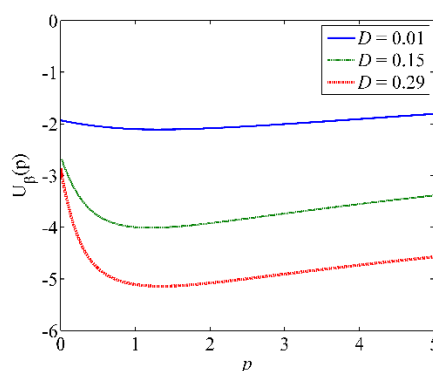


Figure 4. The generalized effective potential as a function of the phosphorus concentration p in the TD-LEE model for the fixed rate of phosphorus intake $c = 1.2$ under different multiplicative noise intensity D , the other parameters are $Q = 0.04$, $\beta = 0.08$, $\lambda = 0.5$, $\tau = 0.2$, $r = 0.55$.

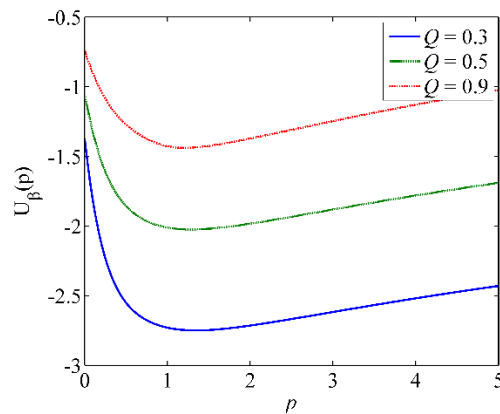


Figure 5. The generalized effective potential as a function of the phosphorus concentration p in the TD-LEE model for the fixed rate of phosphorus intake $c = 1.2$ under different additive noise intensity Q , the other parameters are $D = 2.5$, $\beta = 0.9$, $\lambda = 0.5$, $\tau = 0.08$, $r = 0.55$.

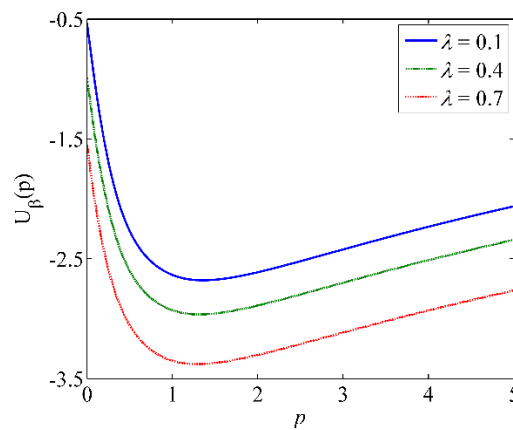


Figure 6. The generalized effective potential as a function of the phosphorus concentration p in the TD-LEE model for the fixed rate of phosphorus intake $c = 1.2$ under different cross-correlation intensity λ , the other parameters are $D = 2.5$, $Q = 0.5$, $\beta = 0.05$, $\tau = 0.2$, $r = 0.55$.

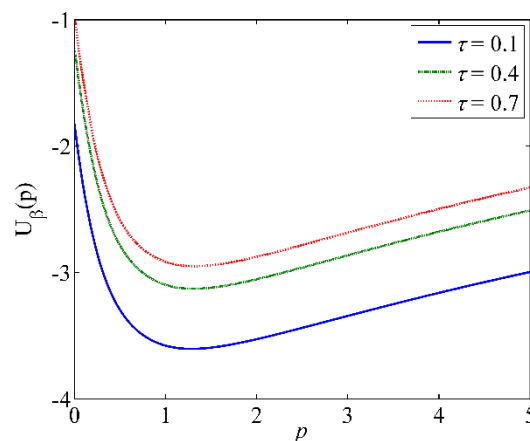


Figure 7. The generalized effective potential as a function of the phosphorus concentration p in the TD-LEE model for the fixed rate of phosphorus intake $c = 1.2$ under different correlation time τ , the other parameters are $D = 2.5$, $Q = 0.5$, $\beta = 0.05$, $\lambda = 0.5$, $r = 0.55$.

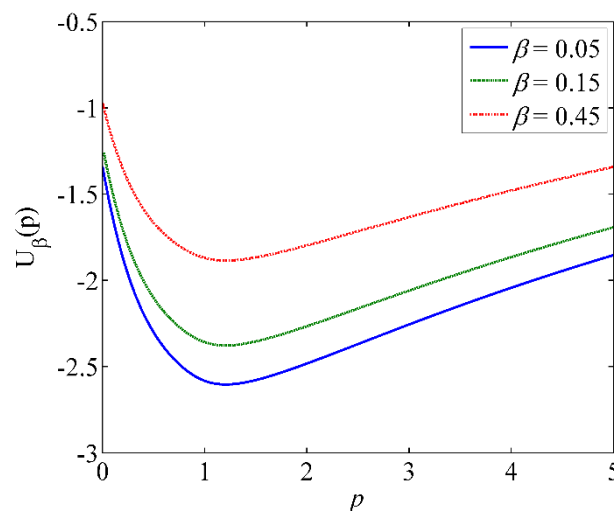


Figure 8. The generalized effective potential as a function of the phosphorus concentration p in the TD-LEE model for the fixed rate of phosphorus intake $c = 1.2$ under different time delay β , the other parameters are $D = 2.5$, $Q = 0.5$, $\tau = 0.08$, $\lambda = 0.5$, $r = 0.55$.

From Fig. 4 to Fig.8, we draw the generalized effective potential as a function of the phosphorus concentration p in the TD-LEE model for the fixed rate of phosphorus intake $c = 1.2$ under different parameters in time delay and colored correlated noises. The influences of the two colored correlated noises and time delay term on the noises-induced oligotrophication will be discussed. The generalized effective potential shows a single valley with different control parameters, which shows the success to excite regime shifts. In Fig. 4, it exhibits the variant trajectories of the generalized effective potential, which switches randomly between the oligotrophic state and the eutrophic one. The depth of potential well become more and more profound with the increasing of multiplicative noise intensity D , then its oligotrophication becomes more pronounced as the values of D increase to relatively large ($D = 0.29$). However, the potential curve tends to be horizontal in the small D case ($D = 0.01$), which indicates the importance of multiplicative noise in the transition of lake trophic state. In summary, it is noting that large enough multiplicative noise intensity is easy to induce oligotrophic state. In Fig. 5, with the increasing of Q , the depth of potential well shows a downward trend and it is the opposite of the multiplicative noise case in Fig. 4. In Fig. 6 and Fig. 7, it is seen that the impact of the cross-correlation intensity and the correlation time between two colored correlated noises on the generalized effective potential is shown. More interestingly, the role of cross-correlation intensity and correlation time is opposite. The incremental cross-correlation intensity can promote the oligotrophication in TD-LEE, while the incremental correlation can destroy the lake ecological environment. In Fig. 8, the effect of time delay term of the TD-LEE model on the noise-induced oligotrophication is also investigated. The equilibrium state be more and more shallow with the increment in β , which demonstrates that the time delay term in certain can damage oligotrophic state in the TD-LEE model.

As we all know, the most effective way to ensure that lake ecosystems remain oligotrophic is to weaken or eliminate alternative eutrophic states. The variation of parameters in the colored correlated noises and time delay term could induce the regime shifts from the eutrophic state to the oligotrophic one, thus enhancing the controllability in lake ecological restoration. Hence, the study of the generalized effective potential in the TD-LEE model is effective to demonstrate the combination of the colored correlated noises and time delay can induce ecology restoration from eutrophic state to oligotrophic state.

5. Conclusion

The influence of colored correlated noises on regime shifts in the time-delayed lake eutrophication ecosystem model, which is used to examine the colored correlated noises-induced oligotrophication effect. What's really interesting is that the time-delayed lake eutrophication ecosystem can exhibit the ecological bistability with the participation of colored correlated noises. In the combination of multiplicative and additive fluctuations in phosphorus cycling process, the time delay and colored correlation will be lead to regime shifts from an equilibrium state to the alternative one. By changing the structure of the potential function with increasing of the appearance of the left potential well, then the eutrophication can be suppressed. The varying parameters of colored correlated noises can not only induce the ecological bistability but also stabilize the oligotrophic state, which reveals that the TD-LEE method can promote the ability of colored correlated noises-induced oligotrophication.

Here, the theoretical basis results with colored correlated noises-induced oligotrophic state in the time-delayed lake eutrophication ecosystem are only explored in this paper. Moreover, we devote ourselves to the study on the generalized effective potential function, where the lake ecosystem model is assumed to be only the paradigmatic lake eutrophication model. It is significative to adopt different evaluation indexes such as mean velocity, residence probability, mean first-passage time and amplitude difference or various complex stochastic systems such as vegetation system, logistic growth system and metapopulation system to investigate rich numerical dynamic results. These clues have been promoting the development of lake eutrophication and stabilizing the ecological environment of lake ecosystem.

Acknowledgments

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