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The Properties and Parameter Estimation of Left-Truncated Skew-Normal Distribution

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Abstract

Regarding the double truncated skew normal distribution proposed by Flecher et al. (2010), we explore a special case when its upper bound tends to infinity -- the left-truncated skew-normal distribution. Specifically, this paper investigates its relevant properties, such as the expressions of the density function, cumulative distribution function, moments, and moment generating function. Additionally, we obtain the expression form of the location-scale extended family of the zero-truncated skew-normal distribution and employ the maximum likelihood method to estimate the parameters of the location-scale extended family distribution. Finally, the performance of the maximum likelihood estimation is evaluated using Monte Carlo simulation.

Keywords

Nonnegative Data; Skew Normal Distribution; Location-Scale Extended Family; Maximum Likelihood Estimation.

1. Introduction

The normal distribution is a common type of distribution in statistics, with its density function curve taking a bell shape and symmetrically distributed around the mean. However, in practical data analysis, we often encounter situations where the symmetry is incomplete. In such cases, by introducing a skewness parameter, the density function curve can be made to no longer be symmetric around the mean, instead shifting to the left or right, thereby better fitting the characteristics of such actual data. Based on this, Azzalini (1985) introduced the skew-normal (SN) distribution. Specifically, if the probability density function (pdf) of a random variable X is:

$$f(x) = 2\varphi(x)\Phi(\alpha x), \tag{1}$$

X is said to follow a standard skew-normal distribution with paramete α, denoted as $X \sim SN(0,1,\alpha)$. The parameter α ($\alpha \in R$) is referred to as the shape parameter, $\varphi(\cdot)$ and $\Phi(\cdot)$ were density and distribution functions of standard normal distribution, respectively. When $\alpha = 0$, the standard skew-normal distribution reduces to the standard normal distribution. When $\alpha \to \pm\infty$, it degenerates into a half-normal distribution. Furthermore, if $Y = \mu + \sigma X$, then Y is said to follow a skew-normal distribution with parameters μ、σ^2、α, where μ, $\sigma > 0$, and α are termed as the location, scale, and shape parameters, respectively. Specifically, when α =0, $Y \sim N(\mu, \sigma^2)$. For further research on this topic, please refer to Henze (1986), Azzalini and Dalla-Valle (1996), Branco and Dey (2001), among others.

Truncated distribution refers to a conditional probability distribution formed by truncating or truncating a certain range. Truncated distributions arise when events occurring above or below a set threshold or beyond a certain range cannot be known or recorded. For example, a survey may require

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respondents within a specific age range to answer questions, resulting in a data distribution within that range. Another example of truncation is economic loss data due to uncontrollable factors. Loss is defined as a negative outcome or cost, with the data being non-negative, leading to the absence of the lower tail of the distribution. Truncation can occur on one or both sides of the range. Truncation occurring on one side of the range is called single truncation, while truncation occurring on both sides of the range is called double truncation. When truncation occurs at the upper end (or right end) of the support range, it is referred to as right truncation. When truncation occurs at the lower end (or left end) of the support range, it is referred to as left truncation. Specifically, when truncation occurs at zero, it is known as zero truncation. Kim (2004) proposed the family of zero-truncated skew-normal distributions and studied their properties. Flecher et al. (2010) derived expressions for the m-th order moments of doubly truncated skew-normal distributions. Lachos et al. (2020) derived scale-mixture moments for truncated skew-normal distributions, extending the results obtained by Kim (2008), Flecher et al. (2010), Genç (2013), and Garay et al. (2017). For truncated versions of other typical distributions, one can refer to the literature such as Cohen Jr. (1949), Thompson (1951), Chapman (1956), Jawitz (2004), and Bebu et al. (2009).

In the relevant research on left-truncated skew-normal distributions, we found that the performance of maximum likelihood estimation has not been studied. Additionally, there is a lack of systematic comparative analysis with the Value at Risk (VaR) calculated under the skew-normal distribution. Therefore, in this paper, we provide explicit expressions for the first four moments of the left-truncated skew-normal distribution, study the maximum likelihood estimation of parameters for the zero-truncated skew-normal distribution, and evaluate the performance of the maximum likelihood estimation for these parameters using Monte-Carlo simulations. Furthermore, we also calculate the actuarial indicator of VaR and find that this distribution exhibits heavier tails compared to the skew-normal distribution.

The organizational structure of this paper is as follows. In Section 2, we review the definition of the left-truncated skew-normal distribution and explore its properties and distribution functions. In Section 3, we derive expressions for its moment-generating function, skewness, and kurtosis. The subsequent sections focus on the zero-truncated skew-normal distribution. Specifically, Section 4 examines its stochastic representation and its location-scale extended expression. Additionally, Section 5 investigates the parameter estimation problem within its location-scale extended distribution. Section 6 derives the risk measure (VaR) and conducts numerical studies.

2. Left-Truncated Skew-Normal Distribution

The definition of the left-truncated skew-normal distribution is specified as follows:

Definition 2.1: We say that the random variable X follows a left-truncated skew-normal distribution with parameters α and s, if it possesses the following density function:

$$f(x;s,\alpha) = C_s(\alpha)\phi(x)\Phi(\alpha x), \qquad x \geq s, s \in R \tag{2}$$

denoted as $X \sim LTSN(0,1,s,\alpha)$. where $C_s(\alpha) = 1\Big/\int_s^{\infty} \phi(u)\Phi(\alpha u)du$.

When specific values are assigned to the parameters s and α, the $LTSN(0,1,s,\alpha)$ reduces to the normal distribution and certain asymmetric normal distributions. Specifically,

Property 2.2 When $\alpha = 0$ and $s \to -\infty$, $LTSN(0,1,s,\alpha)$ degenerate to a standard normal distribution.

Property 2.3 When $s \to -\infty$, $f(x;s,\alpha) \to 2\phi(x)\Phi(\alpha x)$, $LTSN(0,1,s,\alpha)$ degenerate to a skew-normal distribution distribution.

Property 2.4 When $s = 0$, $LTSN(0,1,s,\alpha)$ degenerate to a skew-normal distribution truncated at 0 (see Kim (2004) for details).

ISSN: 2414-1895 DOI: 10.6919/ICJE.202412_10(12).0016

Property 2.5 When $s=0$ and $\alpha \to \infty$, $LTSN(0,1,s,\alpha)$ degenerate to a standard normal distribution truncated at 0.

Property 2.6 If $X \sim LTSN(0,1,s,\alpha)$, let $Y=|X|$, when $s \geq 0$, $X \sim LTSN(0,1,s,\alpha)$.

When $s<0$, $f_Y(y;s,\alpha)=\begin{cases} C_s(\alpha)\phi(y)\Phi(\alpha y), & y \geq -s \\ C_s(\alpha)\phi(y) & , \quad y<-s \end{cases}$.

Property 2.7 Left-truncated skew-normal distribution is unimodal, i.e. $\log f(x;s,\alpha)$ is a concave function of x.

Figures 1 and 2 depict the probability density plots of the $LTSN(0,1,s,\alpha)$ distribution for different values of the parameters α and s.

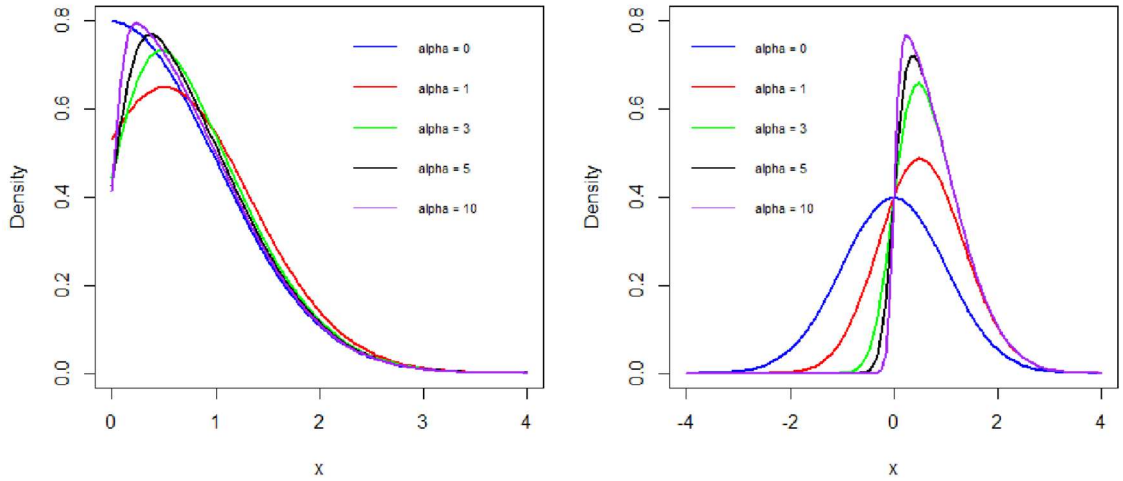


Figure 1. The probability density plot of LTSN distribution at different α values for s= 0 (left), s = -4(right).

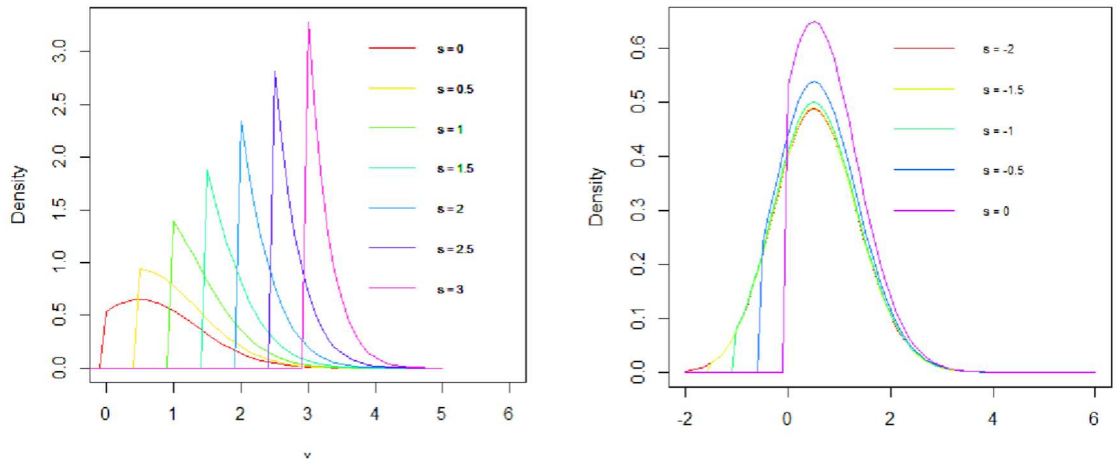


Figure 2. Probability density plot of LTSN distribution at α =1 with s> 0 (left) and s <0 (right).

Definition 2.9 The cumulative distribution function $F(x;s,\alpha)$ of $LTSN(0,1,s,\alpha)$ is:

$$F(x;s,\alpha)=C_s(\alpha)\int_s^x \phi(u)\Phi(\alpha u)du, \qquad x \geq s \tag{3}$$

ISSN: 2414-1895 DOI: 10.6919/ICJE.202412_10(12).0016

Definition 2.10 If the random variable $X \sim LTSN(0,1,s,\alpha)$, and we define $Y = u + \sigma X$, then Y is said to follow a left-truncated skew-normal distribution with parameter μ、σ、s、α denoted as $Y \sim LTSN(\mu,\sigma,s,\alpha)$. The probability density function of Y is given as follows:

$$f(y;\alpha) = \frac{1}{\sigma} C_s(\alpha)\varphi(\frac{y-\mu}{\sigma})\Phi(\alpha\frac{y-\mu}{\sigma}), \quad y \geq s\sigma + \mu, \alpha \geq 0 \tag{4}$$

3. Some Properties of Left-Truncated Skew-Normal Distribution

In this section, we derive some mathematical properties of the $LTSN(0,1,s,\alpha)$ distribution, such as its moments, moment-generating function, skewness, and kurtosis.

Theorem 3.1 (moment generating function) Let $X \sim LTSN(0,1,s,\alpha)$, then its moment generating function is:

$$M_X(t) = C_s(\alpha)\int_s^{\infty} e^{tx}\phi(x)\Phi(\alpha x)dx \tag{5}$$

Theorem 3.2 (moment) Let $X \sim LTSN(0,1,s,\alpha)$, then the n-th moment of X is:

$$E(X^n) = C_s(\alpha)\int_s^{\infty} x^n\phi(x)\Phi(\alpha x)dx \tag{6}$$

Lachos, Garay, and Cabral (2020) provided expressions for the first four moments based on the expectation of random variables defined by their parameters, where the type of distribution that these random variables follow is determined by the values of their parameters. Here, we derive the expressions for the first four moments based on the density function and distribution function of the standard normal distribution. According to Theorem 3.2, we present the first four moments of the $LTSN(0,1,s,\alpha)$ distribution.

Corollary 1: If $X \sim LTSN(0,1,s,\alpha)$, then

$$E(X) = C_s(\alpha)\phi(s)\Phi(\alpha s) + \frac{\alpha}{\sqrt{2\pi}}\frac{1}{\sqrt{1+\alpha^2}}C_s(\alpha)(1-\Phi(s\sqrt{1+\alpha^2})).$$

$$E(X^2) = sC_s(\alpha)\phi(s)\Phi(\alpha s) + \frac{\alpha}{1+\alpha^2}C_s(\alpha)\phi(s)\phi(\alpha s) + 1.$$

$$E(X^3) = s^2C_s(\alpha)\phi(s)\Phi(\alpha s) + 2E(X) + \frac{\alpha s}{1+\alpha^2}C_s(\alpha)\phi(s)\phi(\alpha s) + \frac{\alpha s}{1+\alpha^2}C_s(\alpha)\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{1+\alpha^2}}(1-\Phi(s\sqrt{1+\alpha^2}))$$

$$E(X^4) = s^3C_s(\alpha)\phi(s)\Phi(\alpha s) + 3E(X^2) + \frac{\alpha s^2}{1+\alpha^2}C_s(\alpha)\phi(s)\phi(\alpha s) + \frac{2\alpha}{(1+\alpha^2)^2}C_s(\alpha)\phi(s)\phi(\alpha s)$$

ISSN: 2414-1895 DOI: 10.6919/ICJE.202412_10(12).0016

From Corollary 1, the specific expression for the variance is obtained as follows:

$$Var(X) = C_s(\alpha)\left[s\phi(s)\Phi(\alpha s) + \frac{\alpha}{1+\alpha^2}\phi(s)\phi(\alpha s) \right]^2 - \left[s\phi(s)\Phi(\alpha s)C_s(\alpha) + \frac{\alpha}{1+\alpha^2}\phi(s)\phi(\alpha s)C_s(\alpha) \right]^2 + 1$$

Similarly, specific expressions for skewness and kurtosis can be derived.

4. Special Case: Zero-Truncated Skew-Normal Distribution

Definition 4.1 When the random variable X is truncated at 0, it possesses the following density:

$$f(x;s,\alpha) = C_0(\alpha)\phi(x)\Phi(\alpha x), \quad x \geq 0 \tag{7}$$

denoted as $X \sim LTSN(0,1,0,\alpha)$, where $C_0(\alpha) = 1 \Big/ \int_0^\infty \phi(u)\Phi(\alpha u)du$. This definition is provided by Kim (2004) for the zero-truncated skew-normal distribution.

Based on the stochastic representation of the generalized alpha-skew-normal distribution provided by Sharafi et al. (2017), specifically, we obtain the stochastic representation of the zero-truncated skew-normal distribution by setting $\alpha = 0$, $\lambda = \alpha$, as stated in Theorem 4.2 below.

Theorem 4.2 If $T \sim N(0,1)$ and $W \sim N(0,1)$ are independent, let $X = T \mid \{\alpha T > W\}$ and $T \geq 0$, $W \geq 0$, then $X \sim LTSN(0,1,0,\alpha)$, where $\alpha \geq 0$.

Using the above stochastic representation, we can generate data from $LTSN(0,1,0,\alpha)$ by ”acceptance-rejection” technique by cycling the following steps:

1)Generate random number T from $N(0,1)$.

2)Generate random number W from $N(0,1)$.

3)If $\alpha T > W$ and $T \geq 0$, $W \geq 0$, accept X and deliver $T = X$, otherwise go back to step (1) and continue the process.

Definition 4.3 (location-scale extension of zero-truncated skew-normal distribution) If the random variable $X \sim LTSN(0,1,0,\alpha)$, and we define $Z = \mu + \sigma X$, then Z is said to follow a zero-truncated skew-normal distribution with parameter μ、σ、α denoted as $Z \sim LTSN(\mu,\sigma,0,\alpha)$. The probability density function of Z is given as follows:

$$f(z;\alpha) = \frac{1}{\sigma}C_0(\alpha)\phi\left(\frac{z-\mu}{\sigma}\right)\Phi\left(\alpha\frac{z-\mu}{\sigma}\right), \quad z \geq \mu, \alpha \geq 0 \tag{8}$$

In Figure 3, a probability density function plot of the left-truncated skew-normal distribution when varying at the truncation point s=0 is given.

ISSN: 2414-1895 DOI: 10.6919/ICJE.202412_10(12).0016

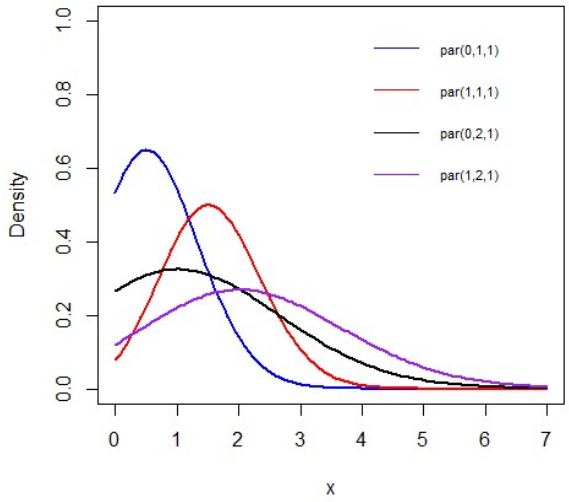


Figure 3. The probability density plot of the LTSN varying in par= (μ, σ, α) at the cutoff point s=0.

5. Maximum Likelihood Estimation and Simulation

In this section, we will employ the maximum likelihood estimation method to estimate the parameters of the left-truncated skew-normal distribution when the truncation point s =0. Additionally, we will utilize Monte Carlo simulation studies to assess the convergence of the maximum likelihood estimators.

5.1 Maximum Likelihood Estimation

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from the $LTSN(\mu,\sigma,0,\alpha)$. Let $\vec{x} = (x_1, x_2, x_3, \ldots, x_n)$ be the observations from pdf (8), the likelihood function can be written as follows:

$$L(\mu,\sigma,\alpha;\vec{x}) = \prod_{i=1}^{n} f(x_i;\mu,\sigma,\alpha) I(x_i \geq \mu) \tag{9}$$

Further, let:

$$\begin{aligned} l(\mu,\sigma,\alpha;\vec{x}) &= \log \prod_{i=1}^{n} f(x_i;\mu,\sigma,\alpha) \\ &= \sum_{i=1}^{n} \log\Phi(\alpha \frac{x_i-\mu}{\sigma}) - \frac{1}{2}\sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma})^2 + n\log C_0(\alpha) - n\log\sigma - \frac{1}{2}n\log(2\pi) \end{aligned} \tag{10}$$

The partial derivatives of (10) with respect to the parameters μ , σ and α are given, respectively, by:

$$\frac{\partial l}{\partial \mu} = -\frac{1}{\sigma}\left[\sum_{i=1}^{n} \alpha h(\alpha \frac{x_i-\mu}{\sigma}) - \sum_{i=1}^{n} \frac{x_i-\mu}{\sigma}\right], \tag{11}$$

$$\frac{\partial l}{\partial \sigma} = -\frac{1}{\sigma}\left[\sum_{i=1}^{n} \alpha(x_i-\mu) h(\alpha \frac{x_i-\mu}{\sigma}) - \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\sigma}\right] - \frac{n}{\sigma}, \tag{12}$$

ISSN: 2414-1895 DOI: 10.6919/ICJE.202412_10(12).0016

$$\frac{\partial l}{\partial \alpha}=\sum_{i=1}^{n}\frac{x_i-\mu}{\sigma}h(\alpha\frac{x_i-\mu}{\sigma})-n\frac{\int_0^{+\infty}u\phi(u)\phi(\alpha u)du}{\left[C_0(\alpha)\right]^3}, \tag{13}$$

where $h(x)=\frac{\phi(x)}{\Phi(x)}$. Equations (11), (12) and (13) are difficult to obtain analytical solutions, which can be obtained by using statistical software.

5.2 Simulation

To evaluate the performance of the maximum likelihood estimation of the left-truncated skew-normal distribution parameter at $s=0$, we perform a comprehensive simulation study, utilizing goodness-of-fit functions and the "$L-BFGS-B$" algorithm in R to obtain Maximum Likelihood Estimates (MLEs). Random number generation algorithms can be used to produce random numbers following an $LTSN(\mu,\sigma,0,\alpha)$ distribution. For sample sizes of 25, 50, ..., 500, we conducted 500 Monte Carlo simulations for each sample size, with the true parameter values set to $\mu=0.3$, $\sigma=3$, and $\alpha=2.1$. The process is carried out as follows:

1)Generate random samples of size n from the $LTSN(\mu,\sigma,0,\alpha)$ distribution, with the repetition of sampling set to 500 times.

2)Based on each sample, compute the maximum likelihood estimates (MLEs) of the parameters.

3)Calculate the sample mean of the maximum likelihood estimates (MLEs). Subsequently, use the following formulas to compute the sample mean squared error (MSE) and absolute bias.

$$MSE(w)=\frac{1}{500}\sum_{i=1}^{500}(\hat{w}_i-w)^2,$$

$$Bias(w)=\frac{1}{500}\sum_{i=1}^{500}(\hat{w}_i-w),\qquad w\in(\mu,\sigma,\alpha)$$

As can be seen from Table 1, as the sample size n increases, the mean of the maximum likelihood estimates of the parameters approaches the true values of the parameters. Additionally, both the standard error and the mean squared error decrease as the sample size n increases.

The simulation results presented in Figures 4 and 5, for various parameter values, demonstrate that the estimated values are quite stable. More importantly, they are close to the true values for these sample sizes. As the sample size n increases, the bias of the estimates decreases. Furthermore, the MSEs decay towards zero as the sample size n increases.

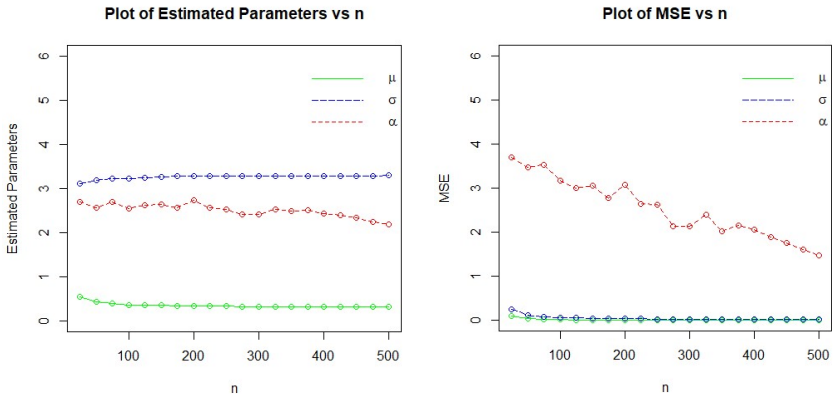


Figure 4. Parameter estimation and sample mean square error for μ = 0.3, σ = 3.3 and α =2.1.

ISSN: 2414-1895 DOI: 10.6919/ICJE.202412_10(12).0016

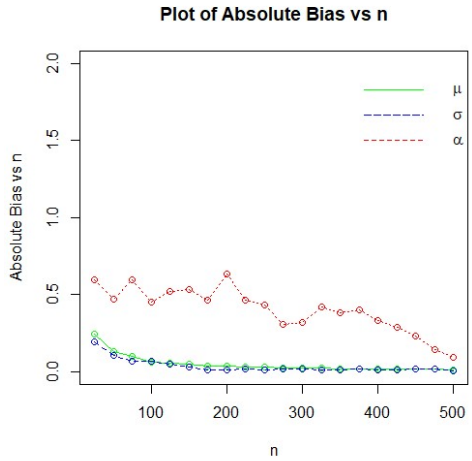


Figure 5. Absolute deviation of μ = 0.3, σ = 3.3 and α =2.1.

Table 1. Numerical simulation of the LTSN distribution.

n	parameters	MLEs	MSE	Absolute Biases
25	μ	0.5423439	0.1008333856	0.24234392
	σ	3.105193	0.24305257	-0.194806873
	α	2.695362	3.703992	0.5953617
50	μ	0.4326835	0.0331322044	0.13268355
	σ	3.195932	0.10990695	-0.104067960
	α	2.567996	3.473194	0.4679956
75	μ	0.3980319	0.0183982904	0.09803186
	σ	3.229715	0.07358989	-0.070284686
	α	2.698557	3.533630	0.5985567
100	μ	0.3626711	0.0078340699	0.06267105
	σ	3.232226	0.05955435	-0.067774413
	α	2.549708	3.160496	0.4497085
200	μ	0.3341325	0.0022076774	0.10441295
	σ	3.289061	0.02893000	0.07158360
	α	2.732447	3.066852	0.05404480
300	μ	0.3248031	0.0011916493	0.02480307
	σ	3.285003	0.02323668	-0.014996728
	α	2.417800	2.134501	0.3178003
400	μ	0.3186937	0.0006695020	0.01869369
	σ	3.286903	0.01729548	-0.013096642
	α	2.433773	2.050159	0.3337725
500	μ	0.3140122	0.0003943709	0.01401223
	σ	3.294040	0.01468824	-0.005959986
	α	2.191684	1.469832	0.0916837

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6. Value at Risk

In this section, we calculate an important risk measure-- value at risk (VaR), which plays a crucial role in portfolio optimization under uncertainty.

6.1 Derivation of VaR Based On LTSN (0,1,0,α)

The mathematical expression for VaR is as follows:

$$\mathrm{P}(X < VaR) = q \tag{14}$$

Where X represents the potential loss from risk over a certain period of time; q represents the quantile. From this, we can see that VaR is actually a point estimate of the distribution function $F(x)$ at the q. $F(x)$ represents the cumulative distribution function of the left-truncated and right-censored normal distribution at the truncation point s =0, i.e.

$$VaR_q(x) = F^{-1}(q) = \inf\{x \geq 0 : F(x) \geq q\}, q \in (0,1) \tag{15}$$

Where $F^{-1}(q)$ is the inverse function of the loss distribution function $F(q)$.

6.2 Numerical Study of the Value of Risk

In this subsection, we present a numerical study on the risk measures of the skew-normal distribution and the $LTSN(\mu,\sigma,0,\alpha)$ distribution. The value at risk (VaR) under the same parameters is compared. The numerical results of the VaR are presented in Table 2 and graphically depicted in Figure 6. A model with a higher value of VaR in risk measurement is regarded as having a heavier tail. The results provided in Table 2 indicate that the proposed model has a higher risk measurement value than the traditional skew-normal model.

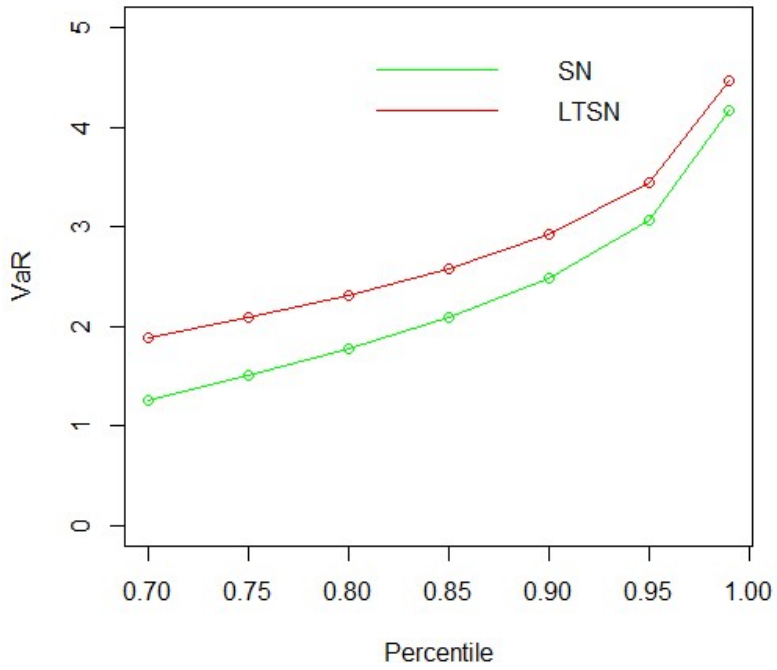


Figure 6. shape=0.11, scale= 1.64, alpha= 0.24.

ISSN: 2414-1895 DOI: 10.6919/ICJE.202412_10(12).0016

Table 2. Numerical studies of VaR.

Distribution	Par	Percentile	VaR
$LTSN(\mu,\sigma,0,\alpha)$	shape=0.11 scale= 1.64 alpha= 0.24	0.7	1.8873
		0.75	2.08274
		0.8	2.30642
		0.85	2.57404
		0.9	2.91962
		0.95	3.44612
		0.99	4.46609
SN	shape=0.11 scale= 1.64 alpha= 0.24	0.7	1.25975
		0.75	1.50172
		0.8	1.77122
		0.85	2.08541
		0.9	2.48083
		0.95	3.06708
		0.99	4.16746

7. Conclusion

This paper conducts an in-depth study of the left truncated skew normal distribution, followed by a discussion of its statistical properties and the derivation of the VaR risk measure for the zero-truncated skew normal distribution. The paper also explores the random number generation algorithm, the numerical study of VaR, and the random simulation problem of parameter estimation for the skew normal distribution. We hope that the left truncated skew normal distribution will find wider application in actuarial science and related fields.

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