The Linear Arboricity of IC-planar Graphs

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Abstract

The linear arboricity $\mathrm{la}(G)$ of a graph G is the minimum number of linear forests that partition the edges of G. In 1981, Akiyama, Exoo and Harary conjectured that $\left\lceil \frac{\Delta(G)}{2} \right\rceil \le \mathrm{la}(G) \le \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ for any simple graph G. A graph G is 1-planar if it can be drawn in the plane so that each edge has at most one crossing. An IC-planar graph is a 1-planar graph satisfying the condition that each vertex is incident with at most one crossing edge. It is shown in this paper that every IC-planar graph G with maximum degree $\Delta \ge 11$ has $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

Keywords

Linear Arboricity; IC-Planar Graph; Linear Coloring; 1-Planar Graph.

1. Introduction

All graphs considered in this paper are simple unless otherwise stated. For a graph G, we use $V(G)$, $E(G)$, $\delta(G)$, and $\Delta(G)$ (for short, Δ) to denote the set of vertices, the set of edges, the minimum degree, and the maximum degree of G, respectively. A *linear forest* is a graph in which each component is a path. A mapping $\phi: E(G) \to \{1,2,\ldots,k\}$ is called a *linear k-coloring* if each color class induces a linear forest. The *linear arboricity*, denoted by $\mathrm{la}(G)$, of a graph G is the minimum number k for which G has a linear k-coloring. This concept is due to Harary [8]. In 1981, Akiyama, Exoo and Harary [1] proposed the following famous **Linear Arboricity Conjecture**:

Conjecture 1: For any graph G, $\left\lceil \frac{\Delta(G)}{2} \right\rceil \le \mathrm{la}(G) \le \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

Conjecture 1 has been confirmed for graphs with $\Delta = 3,4$ in [1,2],for graphs with $\Delta = 5,6,8$ in [5], and for graphs with $\Delta = 10$ in [7]. Suppose that G is a planar graph. Wu [11] first proved that G satisfies Conjecture 1 if $\Delta \ne 7$. The remaining case where $\Delta = 7$ was later settled in [12]. Furthermore,Cygan et al.[4] proved that if $\Delta \ge 9$ then $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$, and conjectured that this is also true for the case of $5 \le \Delta \le 8$.Wu et al.[14] proved that if G is a planar graph without 4-cycles and $\Delta \ge 5$, then $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Wu et al.[15] proved that if G is a planar graph without i-cycles and $\Delta \ge 7$, for $i \in \{4,5\}$, then $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Wu [16] investigated the linear arboricity of out-planar graphs, and showed that every out-planar graph G with $\Delta \ge 3$ has la

$(G) \le \left\lceil \frac{\Delta(G)}{2} \right\rceil$. For a general graph G, Alon [3] proved that, for each $\varepsilon \ge 0$, there exists a constant $C(\varepsilon)$ such that every graph G with $\Delta \ge C(\varepsilon)$ has la $(G) \le (\frac{1}{2}+\varepsilon)\Delta$.Recently, Ferber, Fox, and Jain [6] showed that there exist absolute constants $\eta, C > 0$ such that every graph G has la $(G) \le \frac{\Delta}{2} + C\Delta^{\frac{2}{3}-\eta}$.

For linear arboricity of 1-planar graphs, Zhang et al.[13] proved that if $\Delta \ge 33$ then la $(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Recently, Huang and Jiang [19] improved this result by reducing $\Delta \ge 33$ to $\Delta \ge 25$. Liu et al.[9] proved that every 1-planar graph G has $\mathrm{la}_2(G) \le \left\lceil \frac{\Delta+1}{2} \right\rceil + 14$. This result was recently improved to that $\mathrm{la}_2(G) \le \left\lceil \frac{\Delta+1}{2} \right\rceil + 7$ by Liu et al.[10]. For IC-planar graph, Zhang et al.[17] proved that if $\Delta \ge 17$ then $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Recently, Jiang and Huang [18] improved this result by reducing $\Delta \ge 17$ to $\Delta \ge 15$. In this paper, we improved this result and showed the following result:

Theorem1.1 *Let $M \ge 12$ be a positive integer. If G is an IC-planar graph with $\Delta(G) \le M$, then $\mathrm{la}(G) \le \left\lceil \frac{M}{2} \right\rceil$.*

This leads to the following Corollary 1.2.

Corollary 1.2 *If G is an IC-planar graph with $\Delta \ge 11$, then $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$.*

2. Notations and Preliminaries

To complete the proof of Theorem 1.1, we need to introduce a few concepts and known results. Suppose that G is a plane graph with the face set $F(G)$. For $f \in F(G)$, we use $\partial(f)$ to denote the boundary walk of f and write $f = [u_1u_2...u_n]$ if $u_1, u_2, ..., u_n$ are the vertices of $\partial(f)$ in a cyclic order. Let $V(f) = V(\partial(f))$ and $E(f) = E(\partial(f))$. A vertex of degree k (at most k, at least k, respectively) is called a k-vertex (k^--vertex, k^+-vertex, respectively). Similarly, we can define k-face, k^--face and k^+-face.

Suppose that G is a 1-planar graph which is drawn in the plane such that each edge has at most one crossing and the number of crossings are as few as possible. An IC-planar graph is a 1-planar graph satisfying the condition that each vertex is incident with at most one crossing edge. Let $X(G)$ denote the set of crossings in G. For each $x \in X(G)$, there is a pair of crossing edges $y_1y_2, z_1z_2 \in E(G)$ with x as crossing. Define an *associated plane graph* $G^\times$ as follows.

$V(G^\times) = V(G) \bigcup X(G)$,

$E(G^\times) = E_0(G) \bigcup E_1(G)$,

where $E_0(G)$ is the set of non-crossed edges in G and:

$E_1(G) = \{yx, xz \mid yz \in E(G) \setminus E_0(G)$, and x is a crossing on $yz\}$.

The vertices in $V(G)$ are called *true vertices* of $G^{\times}$, and the vertices in $X(G)$ are called *false vertices* of $G^{\times}$ .Observe that $d_{G^{\times}}(v)=d_G(v)$ if $v\in V(G)$, and $d_{G^{\times}}(v)=4$ if $v\in X(G)$. Let $v\in V(G^{\times})$ be a true vertex. If $vv'\in E(G^{\times})$ and v' is true, then we call v' a *direct-neighbor* of v in $G^{\times}$. If $vw\in E(G^{\times})$ and w is false such that $wv'\in E(G^{\times})$ and $vv'\in E(G)$, then we call v' an *indirect-neighbor* of v in $G^{\times}$. A true vertex is called *small* if $d_G(v)\le 6$ and *big* otherwise. A face $f\in F(G^{\times})$ is called *large* if $d_{G^{\times}}(f)\ge 4$. A 3-face f of $G^{\times}$ is said to be *false* if $V(f)$ contains a false vertex, and otherwise it is *true*. A false 3-face f of $G^{\times}$ is said to be *bad* if $V(f)$ contains a small vertex. For a vertex $v\in V(G^{\times})$ and an integer $i\ge 1$, we use $n_i(v)$, $n_s(v)$, $m_i(v)$, $m_i^{+}(v)$, $m_3^{*}(v)$, $m_3^{b}(v)$ to denote the number of i-vertices, small vertices, i-faces, i^{+}-faces, false 3-faces, and bad false 3-faces which are (direct or indirect) adjacent to v or incident to v, respectively. Liu et al.[20] gave the following useful properties for the associated plane graph of an IC-planar graph.

Lemma 2.1 *Let G be an IC-plane graph and $G^{\times}$ be its associated plane graph. Then the following holds.*

(1) For any two false vertices u and v in $G^{\times}$, $uv\notin G^{\times}$, and every true vertex in $G^{\times}$ is adjacent to at most one false vertex.

(2) If $d_{G^{\times}}(v)=2$, then $m_3^{*}(v)=0$.

(3) If $d_{G^{\times}}(v)=3$, then $m_3^{*}(v)\le 1$.

3. Proof of Theorem 1.1

Suppose that the theorem is false. Let G be a connected minimal counterexample such that every edge is crossed by at most one other edge, every vertex is incident with at most one crossing edge, and crossings are as few as possible.Let $G^{\times}$ be the associated plane graph of G. The proof is divided into four parts as follows. In Part 1, we have collected many useful conclusions for proving the lower bound of linear arboricity of IC-planar graphs. In Part 2, based on the conclusions already obtained in the Part 1, we prove many structural properties of IC-planar graphs. To derive a contradiction on $G^{\times}$, we need to employ the Euler's formula and a discharging method. Both Part 3 and Part 4 are contributed to deal with this long part.

3.1. Part 1. The Known Lemmas

It is easy to know that G is 2-connected and $\delta(G)\ge 2$. Cygan et al.[4] show many structural properties of graph which is not $\left\lceil \frac{\Delta}{2} \right\rceil$-linear colorable. Lemma 3.1 and Lemmas 3.3-3.9 below have been established in [4], these conclusions apply not only to planar graphs, but to all general graphs. The Lemma 3.2 has been established in [18].

Lemma 3.1 *For any edge $uv\in E(G)$, we have $d_G(u)+d_G(v)\ge 2\left\lceil \frac{M}{2} \right\rceil+2$.*

By Lemma 3.1, we can derive that the (direct or indirect) neighbors of 2-vertex are M -vertices, the (direct or indirect) neighbors of 3-vertex are $(M-1)^{+}$ -vertices, the (direct or indirect) neighbors of 4-vertex are $(M-2)^{+}$ -vertices, and small vertices will not adjacent to any small vertices.

Lemma 3.2 *Let $v\in V(G^{\times})$ and $d_{G^{\times}}(v)\ge 7$, then $n_s(v)\le d_{G^{\times}}(v)-\left\lfloor \frac{m_3(v)}{2} \right\rfloor$.*

Lemma 3.3 *Every vertex is at most adjacent to one 2-vertex in* G.

Lemma 3.4 G *does not contain a 2-vertex* v *such that the two neighbors* u *and* z *are not adjacent.*

According to Lemma 3.4, the structural characteristics of IC-planar graph and the request that the crossings are as few as possible, we can get that any 2-vertex will incident with a 3-cycle on which each edge is not a crossing edge in $G^{\times}$. The specific proof process is as follows:

Corollary3.5 *If* $d_{G^{\times}}(v)=2$, *then* v *is incident with a 3-cycle on which each edge is not a crossing edge.*

Proof. Let the two neighbors of v be x, y in G. By Lemma 3.4, the two neighbors of v in G are adjacent in G, it means that $vxyv$ is a 3-cycle in G. If vx is crossed with other edge e and u is a false vertex, then vy, xy are not crossing edges by the definition of IC-planar graph. We can redraw vx, vy to the other side of the xy, so there's one less crossing vertex, has a contradiction. If xy is crossed with other edge e' and z is a false vertex, then vy, vx are not crossing edges by the definition of IC-planar graph. We can redraw xy to the other side of the vx, vy, so there's one less crossing vertex, contradiction. □

Lemma 3.6 G *does not contain a 3-vertex with precisely two pairs of adjacent neighbors, like Fig.1.*

In Fig. 1, vertices marked • have no edges of G incident to them other than those shown, vertices marked ○ may have edges connected to other vertices of G not in the configuration.

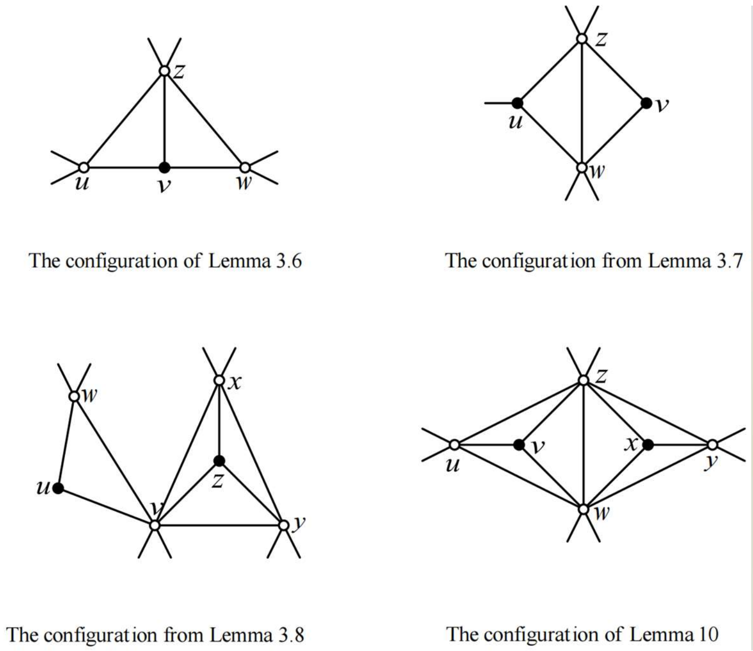

Fig. 1 The configuration of Lemmas.

Lemma 3.7 G *does not contain a 4-cycle* $[vwuz]$ *with* $d_G(v)=2$, $d_G(u)=3$ *and* $wz \in E(G)$,*like Fig.1.*

Lemma 3.8 G *does not contain the corresponding configuration in Fig.1.*

The following Corollary 3.9 can derive by Lemma .3.6-3.8.

Corollary 3.9 *In* G *if a vertex* v *is incident to a triangle with a 2-vertex then every 3-vertex adjacent to* v *is incident to at most one triangle.*

Lemma 3.10 G *does not contain the corresponding configuration in Fig.1.*

The following Corollary 3.11 can derive by lemma3.6 and lemma3.10.

Corollary 3.11 *The two 3-neighbors* x, y *of* v *satisfies that* vx *and* vy *both incident with two 3-cycles in* G, *then* x, y *have only one common neighbor* v.

Lemma 3.12 *If* $\left\lceil \frac{M}{2} \right\rceil \geq 3$ *and* G *contains a vertex* v *of degree at most* $2\left\lceil \frac{M}{2} \right\rceil - 1$ *with two 3-neighbors,then the neighbors of any 3-neighbor of* v *are pairwise nonadjacent.*

3.2. Part 2. Structural Properties

For a k-vertex $v \in V(G^{\times})$, let $v_0, v_1, \ldots, v_{k-1}$ denote the neighbors of v in $G^{\times}$ in a cyclic order, and let $f_0, f_1, \ldots, f_{k-1}$ denote the faces of $G^{\times}$ incident to v with $vv_i, vv_{i+1} \in \partial(f_i)$ for $i = 0, 1, \ldots, k-1$, where the indices are taken as modulo k. If v_i is false, then we assume that v_i is a crossing of the edges vu_i and $x_i y_i$ of G.

Lemma 3.13 *Let* v *be a big vertex, and* $n_3(v) \geq 2$, *then the following conclusions are hold:*

(1) If $d_{G^{\times}}(v) = M - 1$, *then* $m_3(v) \leq d_{G^{\times}}(v) - n_3(v) + 1$;

(2) If $d_{G^{\times}}(v) = M = 13$, *then* $m_3(v) \leq d_{G^{\times}}(v) - n_3(v) + 1$.

Proof. It suffices to show (1) since the proof of (2) is analogous. Since $\left\lceil \frac{M}{2} \right\rceil \geq \left\lceil \frac{12}{2} \right\rceil = 6 \geq 3$ and $2 \times \left\lceil \frac{M}{2} \right\rceil - 1 \geq 2 \times \frac{M}{2} - 1 = M - 1 = d_{G^{\times}}(v)$, so the 3-neighbors of v are not incident with any 3-cycles in G by Lemma 3.12. Suppose that v is not adjacent to any false vertices. Let v_i be a 3-vertex, then f_{i-1}, f_i are 4^+-faces and $m_3(v) \leq d_{G^{\times}}(v) - n_3(v)$. Suppose that v is adjacent to a false vertex v_i. If f_i is a 3-face and v_{i+1} is a 3-vertex, then f_{i-1}, f_{i+1} are 4^+-face by Lemma 3.12 and the definition of IC-planar graph. Thus, $m_3(v) \leq d_{G^{\times}}(v) - n_3(v) + 1$ since u_i may be a 3-vertex. The case that f_{i-1} is a 3-face and v_{i-1} is a 3-vertex can be discussed similarly. If f_{i-1}, f_i are 4^+-faces, or one of f_{i-1}, f_i is a 3-face, say f_i, and v_{i+1} is a 4^+-vertex, then we can infer that $m_3(v) \leq d_{G^{\times}}(v) - n_3(v) + 1$. □

Lemma 3.14 *Suppose that* v *is a big* k*-vertex and* v *is not* adjacent *to any false vertices in* $G^{\times}$, *then the following conclusions are hold:*

(1) If $m_3(v) = k$, *then* $n_3(v) \leq \left\lfloor \frac{k}{3} \right\rfloor$;

(2) If $m_3(v) = k - 1$, *then* $n_3(v) \leq \left\lfloor \frac{k-2}{3} \right\rfloor + 2$.

Proof. (1) We can know that there is a k-cycle $C = [v_0 v_1 \ldots v_{k-1}]$ since $m_3(v) = k$ and v is not adjacent to any false vertices. If v_i is a 3-vertex, then $v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$ are not 3-vertices by Lemma 3.1 and Corollary 3.11. It means that any two 3-vertices on the cycle C have no common neighbors, thus there are at most $\left\lfloor \frac{k}{3} \right\rfloor$ 3-vertices on the cycle C. As a result, $n_3(v) \leq \left\lfloor \frac{k}{3} \right\rfloor$.

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(2) Let f_{k-1} be a 4^+-face, then there is a k-path $P = v_0v_1...v_{k-1}$. Any 3-neighbor $v_i \in \{v_1, v_2, ..., v_{k-2}\}$ of v is incident with two 3-faces, similarly, any two 3-vertices in $\{v_1, v_2, ..., v_{k-2}\}$ have no common neighbors, and there are at most $\left\lfloor \frac{k-2}{3} \right\rfloor$ 3-vertices in $\{v_1, v_2, ..., v_{k-2}\}$. At the same time, v_0, v_{k-1} may be 3-vertex. Therefore, $n_3(v) \le \left\lfloor \frac{k-2}{3} \right\rfloor + 2$. □

Lemma 3.15 *Suppose that v is a big k-vertex and v is adjacent to one false vertex in $G^\times$, then the following conclusion is hold:*

(1) If $m_3(v) = k$, then $n_3(v) \le \left\lfloor \frac{k-1}{3} \right\rfloor + 1$;

(2) If $m_3(v) = k-1$, then $n_3(v) \le \left\lfloor \frac{k}{3} \right\rfloor + 2$.

Proof. (1) Let v_0 be a false vertex, v_0 is crossed by $v_{k-1}v_1$ and vu_0. There is a $k-1$-cycle $C = [v_1...v_{k-1}]$. Then, we can discuss similarly that there are at most $\left\lfloor \frac{k-1}{3} \right\rfloor$ 3-vertices on the cycle C. As a result, $n_3(v) \le \left\lfloor \frac{k-1}{3} \right\rfloor + 1$ since u_0 may be a 3-vertex.

(2) Let f_{k-1} be a 4^+-face. If v_0 or v_{k-1} is a false vertex, then we can discuss similarly with Lemma 3.14(2) that $n_3(v) \le \left\lfloor \frac{k-2}{3} \right\rfloor + 2$. Otherwise, $v_i \in \{v_1, v_2, ..., v_{k-2}\}$ is a false vertex, and there is a $k-1$-path $P = v_0...v_{i-1}v_{i+1}...v_{k-1}$. We can discuss similarly with Lemma 3.14(2) that $n_3(v) \le \left\lfloor \frac{k-3}{3} \right\rfloor$ on $v_i \in \{v_1,,,, v_{i-1}v_{i+1}, ..., v_{k-2}\}$. Therefore, $n_3(v) \le \left\lfloor \frac{k-3}{3} \right\rfloor + 3 = \left\lfloor \frac{k}{3} \right\rfloor + 2$. □

Lemma 3.16 *Suppose that v is a big k-vertex. If $n_2(v) = 1$, then $m_3(v) \le k - \left\lceil \frac{n_3(v)}{2} \right\rceil$.*

Proof. Let v_0 be a 2-vertex. We can infer that f_{k-1} is a 3-face by Corollary 3.5, and f_0 is a 4^+-face, v_{k-1} is a M-vertex. Let's break it down into the following cases. First, assume that f_1 is a 4^+-face or v_1 is a 4^+-vertex. If all of $v_i \in \{v_2, v_3, ..., v_{k-2}\}$ are not false vertices, then $m_4^+(v) \ge \left\lceil \frac{n_3(v)}{2} \right\rceil + 1$ and $m_3(v) \le k - 1 - \left\lceil \frac{n_3(v)}{2} \right\rceil$ by Corollary 3.9. If one of $v_i \in \{v_2, v_3, ..., v_{k-2}\}$ is a false vertex, then u_i may be a 3-vertex and $m_4^+(v) \ge \left\lceil \frac{n_3(v)-1}{2} \right\rceil + 1$. Assume that v_1 is a 3-vertex or a false vertex, then we can discuss similarly that $m_4^+(v) \ge \left\lceil \frac{n_3(v)}{2} \right\rceil$ and $m_3(v) \le k - \left\lceil \frac{n_3(v)}{2} \right\rceil$. □

According to the structural analysis, we can also get the quantitative relationship between the 3-faces and 3-vertices.

Lemma 3.17 *Suppose that v is a big k-vertex. If $n_2(v) = 1$, then $m_3(v) \le 2(k - n_3(v)) - 2$.*

Proof. Let v_0 be a 2-vertex, f_0 is a 3-face and v_1 is a M-vertex. Let the neighbors of v, expect v_0, v_1, the 3-neighbors and the false vertex which v adjacent to, be p-neighbor, and the number of p-neighbors denote by $n_p(v)$.

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Assume that v is not adjacent to any false vertex, then $n_p(v)=k-2-n_3(v)$ by the definition. If f_1 is a 3-face, then v_2 is a p-neighbor of v since v_2 is not a 3-vertex by Lemma 3.8. If f_i is a 3-face and $i\neq 0$, then we can claim that there must be at least one p-neighbor of v in v_i and v_{i+1}. It means that other 3-faces expect f_0 must be at least incident with one p-neighbor of v, thus each p-neighbor of v is at most incident with two 3-faces which v is incident with it. Therefore, $m_3(v)\leq 1+2\times n_p(v)\leq 1+2\times(k-2-n_3(v))=2\times(k-n_3(v))-3$.

Assume that v is adjacent to one false vertex v_i, then $n_p(v)\leq k-2-n_3(v)$ by the definition. We can take out v_i and discuss it again, then $m_3(v)\leq 1+2\times m_p(v)=2(k-1-2-n_3(v))$. When we add v_i to it, then u_i may be a 3-vertex, and a 3-face may be added. Hence, $m_3(v)\leq 1+2\times(k-1-2-(n_3(v)+1))+1=2\times(k-n_3(v))-2$. □

3.3. Part 3. Discharging Rules

To derive a contradiction, we make use of the discharging method.Since G is connected, so is $G^{\times}$. First, by Euler's formula $|V(G^{\times})|+|F(G^{\times})|-|E(G^{\times})|=2$,we can derive the following identity:

$$\sum_{v\in V(G^{\times})}(d_{G^{\times}}(v)-4)+\sum_{f\in F(G^{\times})}(d_{G^{\times}}(f)-4)=-8. \tag{1}$$

Every vertex and face $x\in V(G^{\times})\bigcup F(G^{\times})$ contributes the charge $c(x)=d_{G^{\times}}(x)-4$ to (3.1), so only the charges of 3^{-}-vertices and faces are negative. We define a local redistribution of c''s, preserving their sum, such that the new charge c' (x) is non-negative for all $x\in V(G^{\times})\bigcup F(G^{\times})$. This will contradict the fact that the sum of the new charges is, by (3.1), equal to -8.

We first design several discharging rules as follows.

(*R*1) Let $f=[v_1v_2v_3]$ be a true 3-face with $d_{G^{\times}}(v_1)\leq d_{G^{\times}}(v_2)\leq d_{G^{\times}}(v_3)$ .By $(d_{G^{\times}}(v_1),d_{G^{\times}}(v_2),d_{G^{\times}}(v_2))\rightarrow(a_1,a_2,a_3)$, we mean that the vertex v_i gives f the amount of weight a_i for $i=1,2,3$. We do the following operation:

$$(6^{+},6^{+},6^{+})\rightarrow(\frac{1}{3},\frac{1}{3},\frac{1}{3}),$$

$$(5,9^{+},9^{+})\rightarrow(\frac{1}{5},\frac{2}{5},\frac{2}{5}),$$

$$(4^{-},10^{+},10^{+})\rightarrow(0,\frac{1}{2},\frac{1}{2}).$$

(*R*2) Let $f=[v_1v_2v_3]$ be a false 3-face with v_1 is false and $d_{G^{\times}}(v_2)\leq d_{G^{\times}}(v_3)$. By $(\otimes,d_{G^{\times}}(v_2),d_{G^{\times}}(v_3))\rightarrow(a_1,a_2,a_3)$, we mean that the vertex v_i gives f the amount of weight a_i for $i=2,3$, $\otimes$ express the false vertex. We do the following operation:

$$(\otimes,7^{+},7^{+})\rightarrow(0,\frac{1}{2},\frac{1}{2}),$$

$$(\otimes,6,8^{+})\rightarrow(0,\frac{1}{3},\frac{2}{3}),$$

$$(\otimes,5^{-},9^{+})\rightarrow(0,\frac{1}{5},\frac{4}{5}),$$

$$(\otimes,4^{-},10^{+})\rightarrow(0,0,1).$$

(R3) Each 2-vertex v gets 1 from each (direct or indirect) neighbor of v in $G^{\times}$.

(R4) Each 3-vertex v gets $\frac{1}{3}$ from each (direct or indirect) neighbor of v in $G^{\times}$.

3.4. Part 4. Computation of Weights

Let c' denote the resultant weight function after (R1)-(R4) are carried out on $G^{\times}$. Let us first show that $c'(x)\geq 0$ \$ for all $x\in F(G^{\times})$.Suppose that $f\in F(G^{\times})$. Then $d_{G^{\times}}(f)\geq 3$. If $d_{G^{\times}}(f)=3$, then $c(f)=-1$. If f is true, then $c'(f)=-1+\min\{\frac{1}{2}\times 2,\frac{1}{5}+\frac{2}{5}\times 2,\frac{1}{3}\times 3\}=0$ by (R1).If f is false, then $c'(f)=-1+\min\{\frac{1}{2}\times 2,\frac{1}{3}+\frac{2}{3},\frac{1}{5}+\frac{4}{5},1\}=0$ by (R2). If $d_{G^{\times}}(f)\geq 4$, then $c'(f)=c(f)\geq 0$.

Let us show that $c'(x)\geq 0$ for all $x\in V(G^{\times})$. Suppose that $v\in V(G^{\times})$ is a k-vertex. According to the size of k, we consider several cases as follows.

Case 1. $k=2$.

Then $c(v)=k-4=-2$, and every (direct or indirect) neighbor of v is big by Lemma 3.1. Thus, $c'(v)\geq -2+1\times 2=0$ by (R3).

Case 2. $k=3$.

Then $c(v)=k-4=-1$, and every (direct or indirect) neighbor of v is big by Lemma 3.1. Thus, $c'(v)\geq -1+\frac{1}{3}\times 3=0$ by (R4).

Case 3. $k=4$.

Then $c(v)=0$, and $c'(v)=0$.

Case 4. $k=5$.

Then $c(v)=1$. By (R1)-(R2), v sends $\frac{1}{5}$ to each of its incident 3-faces. Thus, $c'(v)\geq 1-\frac{1}{5}\times 5=0$.

Case 5. $k=6$.

Then $c(v)=2$. By (R1)-(R2), v sends $\frac{1}{3}$ to each of its incident 3-faces.Thus, $c'(v)\geq 2-\frac{1}{3}\times 6=0$.

Case 6. $7\leq k\leq M-5$.

Then $c(v)=k-4$. The (direct or indirect) neighbors of v are 7^{+}-vertices and $m_3^b(v)=0$ by Lemma 3.1 and the definition of bad false 3-face. Thus, by (R1)-(R2), $c'(v)\geq k-4-\frac{1}{3}(m_3(v)-m_3^*(v))-\frac{1}{2}m_3^*(v)\geq k-4-\frac{1}{3}(k-2)-\frac{1}{2}\times 2\geq\frac{1}{3}(2k-13)\geq 0$.

Case 7. $8\leq M-4\leq k\leq M-3$.

Then $c(v)=k-4$. The (direct or indirect) neighbors of v are 5^{+}-vertices and $m_3^b(v)\leq 1$ by Lemma 3.1 and the definition of bad false 3-face. Thus, by (R1)-(R2),

$$\begin{aligned} c'(v) &\geq k-4-\frac{1}{5}(m_3(v)-m_3^*(v))-\frac{1}{2}(m_3^*(v)-m_3^b(v))-\frac{4}{5}\times m_3^b(v) \\ &\geq k-4-\frac{1}{5}(k-2)-\frac{1}{2}\times 1-\frac{4}{5}\times 1 \\ &\geq \frac{1}{10}(6k-45) \\ &\geq \frac{3}{10}\geq 0. \end{aligned}$$

Case 8. $k=M-2\geq 10$.

Then $c(v)=k-4$. The (direct or indirect) neighbors of v are 4^+-vertices and $m_3^b(v)\leq 1$ by Lemma 3.1 and the definition of bad false 3-face.Thus, by (R1)-(R2),

$$\begin{aligned} c'(v) &\geq k-4-\frac{1}{2}(m_3(v)-m_3^b(v))-1\times m_3^b(v) \\ &\geq k-4-\frac{1}{2}(k-1)-1 \\ &\geq \frac{1}{2}(k-9) \\ &\geq 0. \end{aligned}$$

Case 9. $k=M-1\geq 11$.

Then $c(v)=k-4$. The (direct or indirect) neighbors of v are 3^+-vertices and $m_3^b(v)\leq 1$ by Lemma 3.1 and the definition of bad false 3-face. We will discuss the following two cases according to the number of $n_3(v)$.

Case 9.1. $n_3(v)\leq 1$.

By (R4), we should send $\frac{1}{3}$ to the (direct or indirect) 3-neighbor of v in G. Thus, we can compute that:

$$\begin{aligned} c'(v) &\geq k-4-\frac{1}{2}(m_3(v)-m_3^b(v))-m_3^b(v)-\frac{1}{3}n_3(v) \\ &\geq k-4-\frac{1}{2}(k-1)-1-\frac{1}{3} \\ &\geq \frac{k}{2}-\frac{29}{6} \\ &\geq 0. \end{aligned}$$

Case 9.2. $n_3(v)\geq 2$.

By Lemma 3.13, $m_3(v)\leq k-n_3(v)+1$. Then, by (R1)-(R4), we can calculate that:

$$\begin{aligned} c'(v) &\ge k-4-\frac{1}{2}(m_3(v)-m_3^b(v))-1\times m_3^b(v)-\frac{1}{3}n_3(v) \\ &\ge k-4-\frac{1}{2}(k-n_3(v)+1-1)-1-\frac{1}{3}n_3(v) \\ &\ge \frac{1}{2}(k-10)+\frac{1}{6}n_3(v) \\ &\ge 0. \end{aligned}$$

Case 10. $k=M\ge 12$.

Then $c(v)=k-4$. The (direct or indirect) neighbors of v are 2^+-vertices and $n_2(v)\le 1$ by Lemma 3.1 and Lemma 3.3. We will discuss the following two cases according to the number of $n_2(v)$.

Case 10.1. $n_2(v)=0$.

By Lemma 3.2, $n_s(v)\le d_{G^\times}(v)-\left\lfloor \frac{m_3(v)}{2} \right\rfloor$. Thus, $n_3(v)\le k-\left\lfloor \frac{m_3(v)}{2} \right\rfloor$ since $n_2(v)=0$. Therefore, by (R1)-(R4), we can calculate that:

$$\begin{aligned} c'(v) &\ge k-4-\frac{1}{2}(m_3(v)-m_3^b(v))-1\times m_3^b(v)-\frac{1}{3}n_3(v) \\ &\ge k-4-\frac{1}{2}(m_3(v)-1)-1-\frac{1}{3}\times(k-\left\lfloor \frac{m_3(v)}{2} \right\rfloor) \\ &\ge k-4-\frac{1}{2}\times m_3(v)-\frac{1}{2}-\frac{1}{3}\times k+\frac{m_3(v)-1}{6} \\ &\ge \frac{1}{3}\times(2k-14-m_3(v)). \end{aligned}$$

Case 10.1.1. $k=M\ge 14$.

In this case, $c'(v)\ge \frac{1}{3}\times(2k-14-m_3(v))\ge \frac{1}{3}\times(2k-14-k)\ge \frac{1}{3}\times(k-14)\ge 0$.

Case 10.1.2. $k=M=13$.

Note that $c(v)=9$. If $n_3(v)\le 1$, then $c'(v)\ge 9-\frac{1}{2}\times(13-1)-1-\frac{1}{3}\ge 2-\frac{1}{3}=\frac{5}{3}\ge 0$. Otherwise, $n_3(v)\ge 2$, then $m_3(v)\le 13-n_3(v)+1=14-n_3(v)$ by Lemma 3.13. Then, by (R1)-(R4), we can calculate that:

$$\begin{aligned} c'(v) &\ge 9-\frac{1}{2}\times(m_3(v)-m_3^b(v))-1\times m_3^b(v)-\frac{1}{3}\times n_3(v) \\ &\ge 9-\frac{1}{2}\times(14-n_3(v)-1)-m_3^b(v)-\frac{1}{3}\times n_3(v) \\ &\ge \frac{3}{2}+\frac{1}{6}\times n_3(v)\ge 0. \end{aligned}$$

Case 10.1.3. $k=M=12$.

Note that $c(v)=8$. If $m_3(v)\le 10$, then $c'(v)\ge \frac{1}{3}\times(2\times 12-14-m_3(v))\ge \frac{1}{3}\times(10-10)=0$. If $m_3(v)=11$, then $n_3(v)\le \left\lfloor \frac{k}{3}\right\rfloor+2=\left\lfloor \frac{13}{3}\right\rfloor+2=6$ by Lemma 3.14 and 3.15. Then, by (R1)-(R4), we can calculate that:

$$\begin{aligned}c'(v)&\ge 8-\frac{1}{2}\times(m_3(v)-m_3^b(v))-1\times m_3^b(v)-\frac{1}{3}\times n_3(v)\\&\ge 8-\frac{1}{2}\times(11-1)-1-\frac{1}{3}\times 6\\&\ge 8-5-1-2\\&\ge 0.\end{aligned}$$

Suppose that $m_3(v)=12$. $n_3(v)\le \max\{\left\lfloor \frac{k}{3}\right\rfloor,\left\lfloor \frac{k-1}{3}\right\rfloor+1\}=4$ by Lemma 3.14 and 3.15. Then, by (R1)-(R4), we can calculate that:

$$\begin{aligned}c'(v)&\ge 8-\frac{1}{2}\times(m_3(v)-m_3^b(v))-1\times m_3^b(v)-\frac{1}{3}\times n_3(v)\\&\ge 8-\frac{1}{2}\times(12-1)-1-\frac{1}{3}\times 4\\&=\frac{1}{6}\ge 0.\end{aligned}$$

Case 10.2. $n_2(v)=1$.

If $n_3(v)\le 1$, then $c'(v)\ge k-4-\frac{1}{2}(m_3(v)-m_3^b(v))-m_3^b(v)-\frac{1}{3}-1\ge k-\frac{1}{2}(k-1)-1-\frac{16}{3}\ge \frac{k}{2}-\frac{35}{6}\ge \frac{1}{6}$.
Next, we consider the situation that $n_3(v)\ge 2$, and we can know that $n_3(v)\le k-2$, since expect the 2-vertex and the M -vertex in the 3-face which incident with the 2-vertex. By Lemma 3.16, $m_3(v)\le k-\left\lceil \frac{n_3(v)}{2}\right\rceil$. Then, by (R1)-(R4), we can calculate that:

$$\begin{aligned}c'(v)&\ge k-4-\frac{1}{2}(m_3(v)-m_3^b(v))-1\times m_3^b(v)-\frac{1}{3}n_3(v)-1\\&\ge k-4-\frac{1}{2}\times(k-\left\lfloor \frac{n_3(v)}{2}\right\rfloor)-\frac{3}{2}-\frac{1}{3}\times n_3(v)\\&\ge k-4-\frac{k}{2}+\frac{n_3(v)}{4}-\frac{n_3(v)}{3}-\frac{3}{2}\\&\ge \frac{1}{2}\times(k-11)-\frac{1}{12}\times n_3(v)\\&\ge \frac{1}{2}\times(k-11)-\frac{1}{12}\times(k-2)\\&\ge \frac{1}{12}\times(5k-64).\end{aligned}$$

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Case 10.2.1. $k = M \geq 13$.

In this case, $c'(v) \geq \frac{1}{12} \times (5k - 64) \geq \frac{1}{12} \times (65 - 64) = \frac{1}{12} \geq 0$.

Case 10.2.2. $k = M = 12$.

Then $c(v) = 8$, and $n_3(v) \leq 10$ similarly. If $n_3(v) \leq 6$, then:

$$\begin{aligned} c'(v) &\geq \frac{1}{2} \times (12 - 11) - \frac{1}{12} \times n_3(v) \\ &\geq \frac{1}{2} - \frac{1}{12} \times 6 = 0. \end{aligned}$$

Otherwise, $n_3(v) \in \{7,8,9,10\}$. By Lemma 3.17, $m_3(v) \leq 2 \times (k - n_3(v)) - 2 = 22 - 2 \times n_3(v)$. Thus,

$$\begin{aligned} c'(v) &\geq 8 - \frac{1}{2} \times (m_3(v) - m_3^b(v)) - 1 \times m_3^b(v) - \frac{1}{3} \times n_3(v) - 1 \\ &\geq 8 - \frac{1}{2} \times (22 - 2 \times n_3(v) - 1) - 2 - \frac{1}{3} \times n_3(v) \\ &\geq \frac{2}{3} \times n_3(v) - \frac{9}{2} \\ &\geq \frac{1}{6} \geq 0. \end{aligned}$$

This completes the proof of Theorem 1.1. □

4. Conclusion

If G is an IC-planar graph with $\Delta \geq 11$, then $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

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