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Cross-efficiency Evaluation based on Consensus Model

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Abstract

In this paper, three methods are proposed to determine the weight of decision makers, namely deviation maximization based on MPR, information entropy and Fermat point. The cross-efficiency matrix provided by multiple decision makers is converted into MHFPRs. Combined with the weight of decision makers, the consensus model was used to make MHFPR reach a group consensus, so as to evaluate the cross-efficiency of all DMUs. The empirical example shows that the cross-efficiency based on reaching consensus is related to the weight of decision-makers. Three aggregation methods proposed in this paper are different from the traditional methods. The result of information entropy is between deviation maximization based on MPR and Fermat-Torricelli point.

Keywords

CCR; Cross-efficiency; Consensus.

1. Introduction

The CCR model was first proposed by Charnes, Cooper and Rhodes [1]. In the CCR model, the Decision Making Unit (DMU) with an efficiency of 1 is evaluated as effective, and the DMU with an efficiency less than 1 is regarded as invalid. Usually, the results of the CCR model generally lead to multiple DMU being evaluated as effective and cannot be further effectively distinguished. Sexton [2] et al. proposed the cross-efficiency evaluation method by combining self-evaluation efficiency and other-evaluation efficiency, which effectively improved the discrimination ability of each evaluated DMU. Therefore, cross-efficiency has been widely applied.

The cross-efficiency model is constructed by setting a second goal, and the second goals set by different scholars vary from person to person. Doyle and Green [3] proposed benevolent and radical cross-efficiencies. By maximizing and minimizing the evaluation of peers to determine the corresponding weights and cross-efficiency values, all DMU can then be ranked. Wang and Chin [4] proposed the neutral cross-efficiency for cross-efficiency evaluation. By determining the corresponding weights and cross-efficiency values as much as possible to be beneficial to oneself without caring about the thoughts of peers, the corresponding ranking results were given. Shi and Wang [5] defined the ideal frontier and the non-ideal frontier through IDMU and ADMU, and determined the weights by using the deviations from the ideal frontier and the non-ideal frontier to obtain the cross-efficiency values and ranking results. Shi and Chen [6], considering the bounded rational behavior, took the prospect theory based on the interval reference point as the second objective to determine the weights and cross-efficiency values.

At present, a large number of literatures focus on the construction of cross-efficiency models, while there are relatively few literatures related to aggregating the final cross-efficiency. Most of the methods for aggregating the final cross-efficiency are obtained by aggregating the cross-efficiency matrix with the average value, that is, considering that the contribution of each efficiency value is the same. However, this is unreasonable. The contributions of self-assessed efficiency and peer efficiency are different, and they ignore the differences between these efficiencies. Wu and Liang [7] regarded DMU as the participants in the cooperative game, calculated the Shapley value by defining the joint eigenfunction values, and determined the final crossover efficiency in combination with the common weights. Wang and Chin [8] proposed the ordered weighted average operator weights for cross-efficiency aggregation, and allocated reasonable weights for self-evaluation and peer evaluation based on the decision-makers' optimism about the best relative efficiency. Wang and Wu [9] utilized Fermat points to aggregate the interval-crossing efficiency and presented a plant growth algorithm for finding Fermat points. Liu and Chen [10] introduced the psychology of decision-makers by using the theory of regret and aggregated the radical cross-efficiency and the benevolent cross-efficiency respectively, which were consistent with the expectations and outcomes of efficiency in DMU

In group decision-making, each decision-maker expresses their original viewpoints and judges whether consistency is met. If the consistency requirement is not met, they adjust their respective viewpoints to meet the conditions. If the consistency requirement is met, the entire adjusted viewpoint will be the group viewpoint through the corresponding method. The higher the decision-makers' recognition of the group's viewpoints is, the more reasonable the decision-making results will be. Therefore, decision-makers constantly communicate and exchange ideas to form a higher consensus. Initially, in order to make full use of the preference information for pairwise comparisons of the schemes provided by experts, Professor Saaty, an American operations researcher [11], proposed a multi-criteria decision-making scheme combining quantitative and qualitative methods - the Analytic Hierarchy Process (AHP) in the 1970s. Its important feature is to represent the corresponding importance degree levels of the two schemes in the form of the ratio of pairwise importance degrees, thereby constructing the reciprocal Preference Relation (MPR). With the increase in the complexity of decision-making systems, factors such as the limitations of decision-makers' knowledge, the ambiguity of information, and personal preferences, when dealing with practical problems, decision-making information has certain fuzziness and uncertainty. Therefore, various different uncertain preference relationships have been proposed. Orlovsky [12] defined the fuzzy preference relation and introduced the linearity of two kinds of fuzzy relations and studied the equivalence of non-fuzzy and non-dominated schemes. Xu [13] proposed a usage method of the priority of the interval number complementarity judgment matrix based on possibility. Van et al. [14] proposed the triangular fuzzy number preference relation. Xu [15] defines the concepts of intuitionistic preference relation, consistent intuitionistic preference relation, incomplete intuitionistic preference relation and acceptable intuitionistic preference relation. Zhu and Xu [16] considered that the judgments provided by decision-makers could not be aggregated and corrected, defined it as hesitant fuzzy, and proposed a priority ranking method based on multiplicative hesitant fuzzy preference correlation, etc. Based on the preference relationship of each decision-maker, using consistency to judge whether the internal logical thinking of experts is reasonable, having scientific and reasonable consistency plays a crucial role in making correct decisions. Therefore, on the basis of meeting consistency, each fuzzy preference relationship aggregates group viewpoints through corresponding methods and adjusts to reach a consensus.

In this paper, the cross-efficiency matrix is transformed into RE-MHFPR to represent the ambiguous and hesitant preference relationships of each decision-maker, and the cross-efficiency matrices are aggregated based on the process of reaching consensus on the multiplicative hesitant fuzzy preference relationship proposed by Zhang and Wu. By judging and adjusting the RE-MHFPR to meet the acceptable consistency, the deviation maximization, information entropy and Fermat-Torricelli point methods are used to assemble each RE-MHFPR to form a group viewpoint, allowing each decision-maker to communicate and exchange continuously to adjust the group viewpoint and reach a consensus. Finally, all DMU are aggregated and sorted using the score function.

2. Cross-efficiency

2.1. CCR Model

Suppose there are n evaluated DMU, and each DMU consumes m different inputs to produce s different outputs. Let x_{ij} be the i-th input of the j-th and y_{rj} be the r-th output of the j-th.

$$\begin{aligned}
&Maximize \quad \theta_{kk}=\sum\nolimits_{r=1}^{s}u_{rk}y_{rk}\\
&Subject\ to \quad \sum\nolimits_{i=1}^{m}v_{ik}x_{ik}=1\\
&\qquad\qquad \sum\nolimits_{r=1}^{s}u_{rk}y_{rj}-\sum\nolimits_{i=1}^{m}v_{ik}x_{ij}\le 0 \quad j=1,2,\dots,n\\
&\qquad\qquad u_{rk}\ge 0 \quad r=1,2,\dots,s\\
&\qquad\qquad v_{ik}\ge 0 \quad i=1,2,\dots,m
\end{aligned} \tag{1}$$

Among them, θ_{kk} is the CCR efficiency of DMU_K, v_{ik} and u_{rk} are the weights of input and output respectively, and let v_{ik}^* and u_{rk}^* be the optimal solutions of model (1). Obviously, the efficiency of $\theta_{kk}^*=\sum\nolimits_{r=1}^{s}u_{rk}^*y_{rk}$ in Model (1) reflects the self-evaluated efficiency of DMU_K.

2.2. Traditional Cross-efficiency

The optimal input-output weights v_{ik}^* and u_{rk}^* of DMU_K determined in Model (1) are calculated:

$$\theta_{ij}^*=\frac{\sum\nolimits_{r=1}^{s}u_{rk}^*y_{rj}}{\sum\nolimits_{i=1}^{m}v_{ik}^*x_{ij}} \quad (j=1,2,\dots,n) \tag{2}$$

A cross-efficiency of DMU_K against DMU_j can be obtained, reflecting DMU_K against DMU_j (j=1,2,... Peer review of n), that is, the efficiency of other reviews. The self-rated efficiency of DMU_j combined with n-1 cross-efficiencies can obtain the average cross-efficiency of DMU_j, which is specifically shown in Table 1.

Table 1. Cross-efficiency matrix

DMU	Target DMU				Average cross-efficiency
	1	2	$\cdots$	n	
1	θ_{11}	θ_{12}	$\cdots$	θ_{1n}	$\frac{\sum\nolimits_{k=1}^{n}\theta_{1k}}{n}$
2	θ_{21}	θ_{22}	$\cdots$	θ_{2n}	$\frac{\sum\nolimits_{k=1}^{n}\theta_{2k}}{n}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
n	θ_{n1}	θ_{n2}	$\cdots$	θ_{nn}	$\frac{\sum\nolimits_{k=1}^{n}\theta_{nk}}{n}$

Step 1: Trend-Cycle Estimation and Initial Smoothing

Perform trend-cycle analysis to obtain the product A of seasonal variation and random error $\widehat{y}_{m6}$. Apply a centered moving average to the power grid asset data for smoothing:

$$\widehat{y}_{m6} = \frac{(0.5y_{t-6} + y_{t-5} + y_{t-4} + \ldots y_{t+4} + y_{t+5} + 0.5_{yy+6})}{12} \tag{3}$$

For quarterly data, a two-period (e.g., two-quarter) moving average is used, expressed as:

$$\widehat{y}_{q2} = \frac{(0.5y_{t-2} + y_{t-1} + y_{t} + y_{t+1} + 0.5y_{y+2})}{4} \tag{4}$$

The smoothed data $\frac{y}{\hat{y}}$ is then defined as:

$$\frac{y}{\hat{y}} = S \times e \tag{5}$$

where $\hat{y}$ is the normalized seasonal factor, S is the seasonal term, and since seasonality does not exist in the moving data, therefore $\hat{y} = 1$.

Step 2: Eliminate the error term and conduct quarterly analysis on the seasonal component S. The values corresponding to each quarter are called the normalized seasonal factors. After removing the long-term changes, the data A containing the seasonal and random error components is obtained. Normalize each seasonal factor to ensure that the mean of each quarter's indicator equals 1. That is:

$$zb_{im} = \frac{z_i \times 12}{\sum_{j=1}^{12} z_j} \tag{6}$$

Normalization processing of power grid asset data:

$$zb_{iq} = \frac{z_i \times 4}{\sum_{j=1}^{4} z_j} \tag{7}$$

Step 3: Remove the seasonal component from the raw data to obtain seasonally adjusted data.

3. Consensus Models Aggregate Cross-efficiency

3.1. Other Aggregation Methods

Wu, Liang and Yang [6] regarded each DMU as a participant in the cooperative game, and used the Shapley value to construct a model related to its attribution to determine the weight of the aggregated final cross-efficiency value.

The linear programming that selects the worst weights for each DMU through the objective function defines the characteristic function D (S) of the union S, and its imputivity vector $z=\left(z_1, z_2, \cdots, z_n\right)$ satisfies the following individual ideals and union rationality:

Individual: $z_j \geq D(j)$

Alliance: $\sum_{j=1}^{n} z_j = D(N) = 1$

And taking the Shapley value as the representative accountability of the union S, the Shapley value of each DMU is:

$$\varphi_i(D)=\sum_{S|i\in S\subset N}\frac{(s-1)!(n-s)!}{n!}\left\{D(S)-D\left(S\setminus\{i\}\right)\right\}$$

Among them, s is the number of members in the union S.

Based on this, the following is constructed, and the model determines the common weights:

$$\begin{aligned} \min \quad & p \\ \text{subject to} \quad & wE_j^{'}+s_j^{+}-s_j^{-}=z_j \\ & \sum_{i=1}^{n} w_i = 1 \\ & 0\leq s_j^{+}\leq p \\ & 0\leq s_j^{-}\leq p \quad (j=1,2,\cdots,n) \\ & w_i \geq 0 \quad (i=1,2,\cdots,n) \end{aligned}$$

Among them, $E_j^{'}$ represents the j-th column vector of $E^{'}$.

In 2011, Wang and Chin [7] proposed the OWA operator weight aggregation cross-efficiency. The different optimism levels of DM are represented by orness degree, including optimism, pessimism, Laplace and Hurwicz, etc. Different OWA operator weights are provided based on different optimism degrees.

When the orness degree of DM is given, the maximum and minimum parallax are used to determine the weight of the OWA operator:

$$\begin{aligned} \text{minimize} \quad & \delta \\ \text{subject to} \quad & \text{orness}(W)=\alpha=\frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i,\ 0\leq\alpha\leq 1 \\ & \sum_{i=1}^{n} w_i = 1 \\ & w_i - w_{i+1} - \delta \leq 0,\ i=1,\cdots,n-1 \\ & w_i - w_{i+1} - \delta \geq 0,\ i=1,\cdots,n-1 \\ & w_i \geq 0,\ i=1,\cdots,n \end{aligned}$$

The self-evaluated weights and peer weights of the aggregated cross-efficiency are determined by extending the corresponding several situations based on this model.

3.2. Consensus Model

3.2.1. MHFPR is Acceptable for Consistency

In group decision-making, each decision-maker compares the decision-making schemes and provides evaluation preference information. However, there may be several values for the judgment of the decision-making schemes. Therefore, Zhang and Wu [17] proposed MHFPR to represent the ambiguity and hesitation of decision-makers.

Definition 1 On the object set $X=\{x_1, x_2, \cdots, x_n\}$, if the matrix elements satisfy:

$$\begin{cases} h_{ij}^{l} \leq h_{ij}^{l+1}, l=1,2,\cdots,b_{ij}-1 \\ h_{ij}^{r} h_{ji}^{b_{ij}+1-r}=1, r=1,2,\cdots,b_{ij} \\ b_{ij}=b_{ji} \\ h_{ii}=1 \end{cases}$$

Then matrix $\tilde{H}=\{h_{ij}\}_{n\times n}$ is called multiplicative Hesitant Fuzzy Preference Relation (MHFPR).

Among them, $h_{ij}=\left(h_{ij}^{1}, h_{ij}^{2}, \cdots, b_{ij}^{b_{ij}}\right)$ is a multiplicative hesitant fuzzy element (MHFE), representing the preference degree of object x_i over object x_j, and b_{ij} is the number of elements in the multiplicative hesitant fuzzy element (MHFE) h_{ij}.

Definition 2: For two MHFE, $h_1=\left\{\xi_1^{l_1} \middle| l_1=1,2,\cdots,b_1\right\}$ and $h_2=\left\{\xi_2^{l_2} \middle| l_2=1,2,\cdots,b_2\right\}$ ($b_1=b_2$), define operation $h_1 \otimes h_2=\bigcup_{\xi_1^{b_1}\in h_1, \xi_2^{b_2}\in h_2}\left\{\xi_1^{l_1}\times\xi_2^{l_2}\right\}$.

Definition 3 Let h_1 and h_2 be two MHFE, and $\lambda>0$, then $\left(\mathrm{h}_1 \otimes \mathrm{h}_2\right)^{\lambda}=\mathrm{h}_1^{\lambda}\otimes\mathrm{h}_2^{\lambda}$.

Definition 4: For an MHFE $h=\left\{\xi^{l} \middle| l=1,2,\cdots,b\right\}$, $s(h)=\left(\prod_{l=1}^{b}\xi^{l}\right)^{\frac{1}{\#b}}$ is called the scoring function of h. There are two h_1 and h_2. If $s(h_1)>s(h_2)$, then $h_1>h_2$; If $s(h_1)=s(h_2)$, then $h_1=h_2$.

Definition 5: Given an MHFPR h, if $\tilde{h}_{ij}=\underset{k=1}{\overset{n}{\otimes}}\left(h_{ik}^{\frac{1}{n}}\otimes h_{kj}^{\frac{1}{n}}\right)$ $(i,j,k=1,2,\cdots,n)$, then MHFPR is consistent MHFPR $\tilde{h}=\left\{\tilde{h}_{ij}\right\}$.

However, it is impossible for decision-makers to provide exactly the same MHFPR. Therefore, a consistency indicator is proposed to measure whether the MHFPR provided by decision-makers is acceptable, and thus the consistency indicator can be accepted.

Definition 6: For two MHFE $h_1=\left\{\xi_1^{l} \middle| l=1,2,\cdots,b\right\}$ and $h_2=\left\{\xi_2^{l} \middle| l=1,2,\cdots,b\right\}$ ($b_1=b_2$), the deviation between h_1 and h_2 is $d(h_1,h_2)=\frac{1}{bn^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{b}\left(h_{ij}^{l}\times h_{ji}^{l}\right)$.

Definition 7: For an MHFE $h=\left\{\xi^{l}\left|l=1,2,\cdots,b\right.\right\}$ and its consistency matrix $\tilde{h}$, the deviation between the consistency index h of h and the consistency index h_2 is $CI\left(h\right)=d\left(h,\tilde{h}\right)$.

When $CI\left(h\right)\le\overline{CI}$, it is considered that h has acceptable consistency. Otherwise, adjust h. Among them, $\overline{CI}$ is the threshold of the consistency index.

3.2.2. Consensus Measure

When each decision-maker meets the acceptable consistency, it is necessary to aggregate the individual MHFPR to form a group consensus. However, the group consensus is not achieved overnight. It requires continuous communication among decision-makers to adjust the individual MHFPR to meet the consensus threshold. If the consensus is lower than the threshold, the individual MHFPE should be further adjusted to reach a consensus.

In order to aggregate individual MHFPR, the hesitant multiplicative geometric weighting operator (HMWG) is proposed:

Definition 8: There exist n HMFPRS, and the hesitant multiplicative geometric weighting operator is:

$$HMWG(h_1,h_2,\cdots,h_n)=\overset{n}{\underset{i=1}{\otimes}}h_i^{w_i}=\bigcup_{\xi_1^b\in h_1,\xi_2^b\in h_2,\cdots\xi_n^b\in h_n}\left\{\prod_{i=1}^{n}\left(\xi_i^b\right)^{w_i}\right\}$$

Among them, w_i is the weight vector of h_i, that is, the weight vector of the decision-maker.

Definition 9: Let n MHFPRS, $H_k=\left(h_{ij,k}\right)_{n\times n}$ and group MHFPRS $H_c=\left(h_{ij,c}\right)_{n\times n}$ be defined. Then, the consensus degree index of H_k is the deviation between H_k and H_c, $GCI\left(H_k\right)=d\left(H_k,H_c\right)$.

When $GCI\left(H_k\right)\le\overline{GCI}$, a consensus is reached. Otherwise, adjust h to continue aggregating the group consensus. Among them, $\overline{GCI}$ is the threshold of the consensus degree.

3.2.3. Consensus Measure

Each decision-maker provides an MHFPR. When all MHFPRS are aggregated to represent a group consensus, the weight of each decision-maker is unknown. Usually, the decision-maker weights are given subjectively or determined through subjective methods. In this paper, three methods, namely the maximization of deviation combining subjectivity and objectivity, objective information entropy, and Fermat-Torricelli point, are used to determine the weights of each MHFPR.

4. Aggregate the Cross-efficiency based on RE-MHFPR

4.1. Maximization of Deviation

Deviation maximization is to determine the attribute weights based on the differences between attributes. If the difference in attribute values is small, it indicates that the attribute has a small influence on the ranking of the scheme. If the attribute values vary greatly, it indicates that the attribute has a significant impact on the sorting of the scheme. Ma Yonghong and Zhou Rongxi [17] first used the deviation maximization model to determine the weight of each attribute, and then used deviation maximization to determine the weight of the decision-maker.

However, in this paper, subjective methods are used to determine the weights of each attribute, and objective methods are used to determine the weights of decision-makers. The MHFPR of each decision-maker is degraded into the multiplicative preference relation (MPR) through the score function, and the weight vector is obtained by using the logarithmic least square method to determine the weight of each attribute, calculate the comprehensive attribute value of each scheme, and thereby determine the weight of the decision-maker based on the maximization of deviations.

Using Definition 4, degenerate MHFPR to MPR $A=\left(a_{ij}\right)$. The weight vector of MPR is determined by using the logarithmic least square method, which is defined as the objective function of the following optimization problem [18] :

$$\begin{aligned} \min \quad & \sum_{i=1}^{n}\sum\nolimits_{i>j}\left[\ln\left(a_{ij}\right)-\left(\ln\left(\beta_i\right)-\ln\left(\beta_j\right)\right)^2\right] \\ \textit{subject to} \quad & \sum_{i=1}^{n}\beta_i=1 \\ & \beta_i\geq 0 \end{aligned}$$

Crawford and Williams discovered that its unique solution can be represented by the row geometric mean of MPR:

$$\beta_i=\frac{\sqrt[\frac{1}{n}]{\prod_{j=1}^{n}s_{ij}}}{\sum_{i=1}^{n}\left(\sqrt[\frac{1}{n}]{\prod_{j=1}^{n}s_{ij}}\right)}$$

Determine the weights of each attribute based on the row geometric mean, and use this weight to obtain the deviation $V_{ij}\left(\beta\right)$ between scheme x_i and all other schemes under each decision-maker's decision:

$$V_{ij}\left(\beta\right)=\sum_{r=1}^{n}\left(s_{ij}\beta_j-s_{rj}\beta_j\right)\left(i=1,2,\cdots,n\ j\right)$$

Based on maximizing the total deviation of each decision-maker, obtain the decision-maker weights:

$$\begin{aligned} \max \quad & \sum_{k=1}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|s_{ik}\beta_{ik}-s_{jk}\beta_{jk}\right|\lambda_k \\ \textit{subject to} \quad & \sum_{k=1}^{K}\lambda_k^2=1 \\ & 0\leq\lambda_k\leq 1\ \left(\lambda=1,2,\cdots,K\right) \end{aligned}$$

Among them, λ_k is the weight of the KTH decision-maker.

4.2. Information Entropy

Information entropy is the probability of occurrence of a certain specific piece of information, mainly a measure of information. The smaller the probability of an event occurring, the greater the amount of information it will generate; the larger the probability of an event occurring, the smaller the amount of information it will generate. Information entropy only gives a numerical value in terms of quantity and does not indicate the importance of the information. Wu and Sun[19] calculated the final cross-efficiency of weight aggregation using information entropy. Cao Linjian et al. [20] utilized information entropy to distinguish the DMU evaluated as effective by the CCR model. Hu Qinghong and Wang Yingming [21] aggregated the cross-efficiency values proposed by various scholars using information entropy. However, in this paper, information entropy is utilized to extract the amount of preference information of each decision-maker, thereby objectively obtaining the weights of the decision-makers.

The steps for extracting decision-makers' preference information using information entropy are as follows:

Step1: Calculate the proportion of the i-th plan under the j-th decision-maker to that decision-maker:

$$p_{ij} = \frac{x_{ij}}{\sum_{i=1}^{n} x_{ij}} \quad \left(i = 1, 2, \cdots, n \quad j = 1, 2, \cdots, K\right)$$

Step2 Calculate the entropy value of the j-th decision-maker:

$$\alpha_j = -c \sum_{i=1}^{n} p_{ij} \ln\left(p_{ij}\right) \quad \left(j = 1, 2, \cdots, K\right)$$

Among them, $c = \dfrac{1}{\ln\left(n\right)} > 0$

Step3 Calculate the redundancy (difference) of information entropy:

$$q_j = 1 - \alpha_j \quad \left(j = 1, 2, \cdots, K\right)$$

Step4 Calculate the weights of each decision-maker:

$$\lambda_j = \frac{q_j}{\sum_{j=1}^{K} q_j}$$

4.3. Determine the Decision-maker Weights based on Fermat-Torricelli Point

In early 1643, Fermat proposed an extreme value problem in a letter to the Italian scholar Torricelli: Suppose there are n random points on a plane, and try to find the only point that minimates the sum of the Euclidean distances to these n points. This problem was first solved by Torricelli, so this point is called Fermat-Torricelli Point.

For the research on the Fermat-Torricelli point, most of the literature focuses on the study of how to find this point, and part of it is the research on the application of the Fermat-Torricelli

Point in mathematics. However, Wang and Wu applied the Fermat-Torricelli point from a new perspective. They selected interval data and obtained the interval cross-efficiency matrix using the upper and lower bound values respectively, regarding each interval as a two-dimensional coordinate point. The Fermat-Torricelli point is found by using the plant growth algorithm to aggregate the interval-crossing efficiency and rank it.

Definition 10: Suppose there are n points in a two-dimensional space plane. If there exists a unique point P^* that satisfies the Euclidean distances of the other given points:

$$D = \min \sum_{i=1}^{n} \left| P^* P_l \right| = \min \left(\sqrt{\left(x^* - a_1\right)^2 + \left(y^* - b_1\right)^2} + \cdots + \sqrt{\left(x^* - a_n\right)^2 + \left(y^* - b_n\right)^2} \right)$$

Then P^* is the Fermat-Torricelli point.

This paper also employs the Fermat-Torricelli point from another perspective, using this point to aggregate the viewpoints of various decision-makers, and by maximizing its similarity to the weighted aggregation method, the optimal decision-maker weights are achieved. Suppose the score function of MHFPR for each decision-maker is mapped to a point y_k in the n-dimensional space. The following model is constructed based on the definition of Fermat-Torricelli Point to find the point $Y^*(y_1, y_2, \cdots, y_n)$ that satisfies Pareto optimality:

$$\min \sum_{k=1}^{K} \sqrt{\sum_{i=1}^{n} \left(s_{ik} - y_{ik}\right)^2}$$

Based on this aggregation of the information Y^* of each decision-maker, it objectively reflects the comprehensive views of each decision-maker on all DMU.

The comprehensive score $R = \left(r_i\right) = \left(\sum_{i=1}^{n} w_k s_{ik}\right)$ of each DMU can be weighted and aggregated by using the decision-maker weight. Similar to the Fermat-Torricelli point, it is to extract the decision-maker information and aggregate all DMU to obtain the comprehensive opinion.

Therefore, there is a similarity between the comprehensive information Y and R obtained by the weighted average of Fermat-Torricelli point and the decision-maker weights. Here, the decision-maker weights will be determined by maximizing the similarity between the two.

For each DMU, the similarity represented by P_i:

$$P_i = \frac{COV\left(r_i, y_i\right)}{\sigma_{r_i} \times \sigma_{y_i}} \quad \left(i = 1, 2, \cdots, n\right)$$

Based on maximizing the total similarity of each DMU, obtain the decision-maker weights:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} \frac{COV\left(r_i, y_i\right)}{\sigma_{r_i} \times \sigma_{y_i}} \\ \textit{subject to} \quad & \sum_{k=1}^{K} \lambda_k = 1 \\ & 0 \le \lambda_k \le 1 \qquad k = 1, 2, \cdots, K \end{aligned}$$

Among them, λ_k is the weight of the k-th decision-maker.

4.4. Construct MHFPR based on Cross-efficiency

The cross-efficiency matrix is transformed into RE-MHFPR through the method proposed by Meng and Xiong[22], that is, the ratio of the self-rated efficiency of each pair of decision-making units to the peer efficiency:

$$\mathrm{E}=\begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{pmatrix}$$

Among them,

$$\begin{cases} e_{ij}=\left\{\dfrac{\theta_{ii}}{\theta_{jj}},\dfrac{\theta_{ii}}{\theta_{ji}},\dfrac{\theta_{ij}}{\theta_{jj}},\dfrac{\theta_{ij}}{\theta_{ji}}\right\}, & i,j=1,2,\cdots,n,\ i\neq j \\ e_{ii}=\left\{\dfrac{\theta_{ii}}{\theta_{ii}}\right\}=\{1\}, & i=1,2,\cdots,n \end{cases} \tag{8}$$

The cross-efficiency matrices calculated by each scholar were converted into RE-MHFPR using the above formula, and based on the above description, individual RE-MHFPR was adjusted to reach a group consensus. After that, all decision units are ranked through the score function.

4.5. Algorithm

The initial iteration steps are $l=0$, the acceptable consistency threshold is $\overline{CI}=1.05$, and the group consensus thresholds are $\overline{GCI}=1.01$, $\delta\in(0,1)$, and $\eta\in(0,1)$:

Step 1: Through formula (6), convert each CEM to RE-MHFPR h_{ij};

Step 2 utilizes the algorithms of Definition 1 and Theorem 1 to obtain the consistency MHFPR $\tilde{h}_{ij}$ of h_{ij};

Step 3: Through the definition of $CI(h)=d(h,\tilde{h})$, calculate the CI value of each MHFPR. If it is $CI(h)\leq\overline{CI}$, then perform Step 5. Otherwise, proceed to the next step;

Step 4 Set $K^{(l)}=\left\{k\in 1,2,,\cdots,n \middle| CI(h)>\overline{CI}\right\}$, then adjust MHFPR $H_k^{(l)}=\left(h_{ij,k}^{(l)}\right)$ through the following formula

$$h_{ij,k}^{l+1}=\begin{cases}\left(h_{ij,k}^{l}\right)^{\delta}\otimes\left(\tilde{h}_{ij,k}^{l}\right)^{(1-\delta)}, & k\in K^{(L)} \\ h_{ij,k}^{(l)}, & k\notin K^{(l)}\end{cases}$$

Iterate the number of steps $l=l+1$. After that, execute Step 2.

Step 5: Determine the decision-maker weight λ_α by using three methods: deviation maximization, information entropy, and Fermat point.

Step 6: Based on the weights of each decision-maker, and using HMWG to aggregate all individual MHFPRS to obtain group $H_c^{(l)} = \left(h_{ij,c}^{(l)}\right)$;

Step 7: Calculate the group consensus indicator $GCI\left(H_k^{(l)}\right) = d\left(H_k^{(l)}, H_c^{(l)}\right)$. If it is $\overline{GCI} \le 1.01$, then perform Step 9. Otherwise, proceed to the next step;

Step 8 When $\overline{GCI} > 1.01$, adjust MHFPR $H_k^{(l+1)} = \left(h_{ij,k}^{(l+1)}\right)$ through the following formula

$$H_k^{(l+1)} = \left(h_{ij,k}^{(l)}\right)^{\eta} \otimes \left(\tilde{h}_{ij,k}^{(l)}\right)^{(1-\eta)}$$

Iterate step $l = l + 1$. After that, execute step 6.

Step 9 uses HMWG to aggregate all the preference information $h_{i,c}^{(l)} = \overset{n}{\underset{j=1}{\otimes}} \left(h_{ij,c}^{(l)}\right)^{\frac{1}{n}}$ $(i = 1, 2, \cdots, n)$ of row i of $H_c^{(l)}$ and obtain all the preference degrees of $h_{i,c}^{(l)}$ of decision unit x_i;

Step 10 Calculate the score function $s\left(h_{i,c}^{(l)}\right)(i = 1, 2, \cdots, n)$ of $h_{i,c}^{(l)}$, and its formula is as follows:

$$s\left(h_{i,c}^{(l)}\right) = \left(\prod_{s=1}^{p} \left(h_{i,c}^{(l)}\right)^{b}\right)^{\frac{1}{p}};$$

Step 11 sorts the score values of each decision unit calculated based on the score function in Step 10.

5. Empirical Analysis

The cross-efficiency evaluation analysis of the seven academic departments (DMU) of a university was conducted using Wang and Chin, and the input-output data are shown in Table 2. The last column of Table 2 shows the CCR efficiencies of the seven departments calculated by Model (1). It can be seen that the CCR of 7 passenger airlines is invalid, and the efficiency of the remaining 7 passenger airlines is 1, making it impossible to further distinguish.

Table 2. Input-output data and CCR efficiency values of seven academic departments

DMU	Input			Output			CCR
	x_1	x_2	x_3	y_1	y_2	y_3	
DMU_1	12	400	20	60	35	17	1
DMU_2	19	750	70	139	41	40	1
DMU_3	42	1500	70	225	68	75	1
DMU_4	15	600	100	90	12	17	0.8197
DMU_5	45	2000	250	253	145	130	1
DMU_6	19	730	50	132	45	45	1
DMU_7	41	2350	600	305	159	97	1

5.1. Consistency Analysis

According to the algorithm in Section 4.4, the consensus model is used to assemble the cross-efficiency matrices of the five models: radical (H_1), benevolent (H_2), neutral (H_3), prospect theory-based (H_4), and both (H_5) in this paper.

Using formula to convert H_1, H_2, H_3, H_4 and H_5 to their respective RE-MHFPR, then using formulas to calculate the corresponding consistency matrices, thus obtaining: $CI(H_1)=1.0504$, $CI(H_2)=1.0131$, $CI(H_3)=1.0278$, $CI(H_4)=1.0536$, according to CI formulas. The threshold selected in this paper is 1.05. Obviously, the cross efficiency of the radical type and the prospect theor-based one does not meet the consistency requirements. Therefore, the preference relationship between the two decision-makers is adjusted according to formula, which is,. At this point, all RE-MHFPR satisfy acceptable consistency.

The five RE-MHFPR were adjusted through formula to all satisfy the acceptable consistency. Then, the three methods, namely, maximization of deviation based on MPR, information entropy, and Fermat-Torricelli point, were used to determine the weights of three different decision-makers. Based on these three different groups of situations, RE-MHFPR has reached a group consensus.

5.2. The Process of Reaching a Group Consensus

According to Part 4.2.4.1, the decision-maker weights are determined through the maximization of deviations based on MPR. The consensus matrix is assembled using step 6 in part 4.4, and then the acceptable consensus degrees of each RE-MHFPR and the consensus matrix are determined, which are $GCI(H_1)=1.0776$, $GCI(H_2)=1.0437$, $GCI(H_3)=1.0516$, $GCI(H_4)=1.1145$, and $GCI(H_5)=1.0602$ respectively. The threshold selected in this paper is 1.01. Obviously, None of the five RE-MHFPR satisfy the acceptable consensus degree. Therefore, the preference relationship of the decision-makers is adjusted according to formula (4-20), namely $GCI(H_1)=1.0007$, $GCI(H_2)=1.0004$, $GCI(H_3)=1.0005$, $GCI(H_4)=1.0011$, $GCI(H_5)=1.0006$. At this time, all RE-MHFPR satisfy the acceptable consensus degree.

Secondly, the decision-maker weights are determined through information entropy. The consensus matrix is assembled using step 9, and then the acceptable consensus degrees of each RE-MHFPR and the consensus matrix are determined, which are $GCI(H_1)=1.1052$, $GCI(H_2)=1.0233$, $GCI(H_3)=1.0330$, $GCI(H_4)=1.1245$, and $GCI(H_5)=1.0412$ respectively. The threshold selected in this paper is 1.01. Obviously, none of the five RE-MHFPR satisfy the acceptable consensus degree. Therefore, Adjust the preference relationship of decision-makers according to the formula, namely $GCI(H_1)=1.0010$, $GCI(H_2)=1.0002$, $GCI(H_3)=1.0003$, $GCI(H_4)=1.0012$, $GCI(H_5)=1.0004$. At this time, all RE-MHFPR satisfy the acceptable consensus degree. Table 3 shows the acceptable comatrices based on information entropy.

Then, the decision-maker weights are determined through Fermat-Torricelli point. The consensus matrices are assembled using the previous chapters, and then the acceptable consensus degrees of each RE-MHFPR and the consensus matrix are determined, which are $GCI(H_1)=1.1274$, $GCI(H_2)=1.0060$, $GCI(H_3)=1.0154$, $GCI(H_4)=1.1686$, and $GCI(H_5)=1.0523$ respectively. The threshold selected in this paper is 1.01. Obviously, only the benevolent RE-MHFPR satisfies the acceptable consensus degree. Therefore, by adjusting the preference relationships of the remaining four decision-makers according to the formula, namely $GCI(H_1)=1.0032$, $GCI(H_3)=1.0010$, $GCI(H_4)=1.0051$, and $GCI(H_5)=1.0030$, all

RE-MHFPR satisfy the acceptable consensus degree. Table 4 shows the acceptable consensus matrix based on Fermat points.

Table 3. Consensus matrix based on information entropy

Consensus matrix based on information entropy						
[1,1,1]	[0.729,0.816,1.233,1.380]	[0.785,0.787,1.285,1.288]	[0.569,0.704,1.512,1.872]	[0.708,0.789,1.381,1.539]	[0.729,0.807,1.307,1.447]	[0.587,0.689,1.433,1.683]
[1.372,1.226,0.811,0.725]	[1,1,1,1]	[0.889,0.988,0.967,1.075]	[0.994,1.092,1.222,1.342]	[0.794,0.925,1.049,1.222]	[0.922,0.954,1.042,1.078]	[0.721,0.840,1.203,1.400]
[1.273,1.271,0.778,0.776]	[1.125,1.012,1.034,0.930]	[1,1,1,1]	[0.756,0.943,1.604,2.000]	[0.575,0.820,1.347,1.920]	[0.773,0.845,1.156,1.263]	[0.456,0.690,1.975,2.989]
[1.759,1.421,0.661,0.534]	[1.006,0.916,0.819,0.745]	[1.322,1.060,0.623,0.500]	[1,1,1,1]	[0.576,0.676,0.998,1.171]	[0.730,0.758,0.951,0.988]	[0.581,0.660,1.262,1.434]
[1.412,1.267,0.724,0.650]	[1.259,1.081,0.954,0.819]	[1.740,1.220,0.743,0.521]	[1.737,1.479,1.002,0.854]	[1,1,1,1]	[1.0213,0.991,1.037,1.006]	[0.818,0.862,1.088,1.146]
[1.372,1.239,0.765,0.691]	[1.085,1.049,0.960,0.928]	[1.293,1.184,0.865,0.792]	[1.371,1.320,1.051,1.012]	[0.979,1.009,0.964,0.994]	[1,1,1,1]	[0.760,0.859,1.152,1.302]
[1.705,1.451,0.698,0.594]	[1.386,1.191,0.831,0.714]	[2.194,1.450,0.506,0.335]	[1.721,1.515,0.792,0.698]	[1.222,1.160,0.919,0.872]	[1.316,1.165,0.868,0.768]	[1,1,1,1]

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Table 4. Consensus matrix based on Fermat-Torricelli point

Consensus matrix based on Fermat-Torricelli point						
[1,1,1,1]	[0.865,0.911,1.044,1.098]	[0.931,0.932,1.286,1.288]	[0.745,0.945,1.271,1.613]	[0.922,0.946,1.131,1.160]	[0.924,0.952,1.064,1.096]	[0.883,0.917,1.084,1.124]
[1.156,1.098,0.958,0.910]	[1,1,1,1]	[0.848,0.988,1.067,1.242]	[1.147,1.178,1.362,1.398]	[0.836,0.939,1.012,1.136]	[0.968,0.976,1.011,1.019]	[0.917,0.947,1.0518,1.087]
[1.075,1.073,0.778,0.776]	[1.179,1.0119,0.938,0.805]	[1,1,1,1]	[0.850,1.128,1.627,2.160]	[0.638,0.946,1.350,2.004]	[0.774,0.795,1.039,1.067]	[0.666,0.904,2.438,3.310]
[1.343,1.058,0.787,0.620]	[0.871,0.849,0.734,0.715]	[1.177,0.887,0.614,0.463]	[1,1,1,1]	[0.453,0.576,0.873,1.108]	[0.662,0.697,0.859,0.904]	[0.631,0.653,0.937,0.969]
[1.084,1.057,0.884,0.862]	[1.196,1.065,0.988,0.880]	[1.568,1.057,0.741,0.500]	[2.205,1.74,1.146,0.902]	[1,1,1,1]	[0.958,0.997,1.035,1.076]	[0.912,0.922,1.020,1.031]
[1.083,1.051,0.940,0.912]	[1.033,1.025,1.000,0.982]	[1.292,1.259,0.962,0.938]	[1.510,1.435,1.164,1.106]	[1.043,1.003,0.966,0.929]	[1,1,1,1]	[0.939,0.965,1.037,1.066]
[1.132,1.091,0.923,0.890]	[1.091,1.056,0.951,0.920]	[1.502,1.106,0.410,0.302]	[1.584,1.532,1.067,1.032]	[1.097,1.084,0.981,0.970]	[1.065,1.036,0.965,0.939]	[1,1,1,1]

Finally, the score values of the three acceptable consensus matrices are calculated based on steps 11 and 12 of the algorithm. The final ranking is obtained from the score values of each RE-MHFPR, as shown in columns 2 to 4 of Table 5.

Table 5. The comparison of the assembly methods in this article with the traditional ones

DMU	Deviation maximization based on MPR	Information entropy	Fermat Point	Meanwhile based on both	aggressive	Benevolent	Neutral	Based on prospect theory
1	1	4	5	5	1	3	2	2
2	6	3	3	4	4	2	3	4
3	2	1	1	6	3	6	6	3
4	7	7	7	7	7	7	7	7
5	5	5	4	3	5	5	5	5
6	4	2	2	1	2	1	1	1
7	3	6	6	2	6	4	4	6

It can be known from Table 5 that there are differences among the results of the three aggregation methods in this paper, indicating that the final ranking result is related to the decision-maker's weight, and all three aggregation methods in this paper are referential.

Compared with the traditional aggregation methods, except for the last DMU_4, the sorting results of the other DMU are basically different, and the deviation maximization based on MPR is significantly different from the other aggregation results. This indicates that the aggregation method in this paper is also a reasonable and effective method for aggregating the cross-efficiency matrix.

6. Conclusion

Cross-efficiency is a common method for differentiating DMU. In the past, most of the research was on the calculation of cross-efficiency, and the average value was used to aggregate the final cross-efficiency. In recent years, many scholars have conducted research on the aggregated cross-efficiency and aggregated the cross-efficiency matrix by using methods such as the OWA operator, Fermat point or regret theory. In this paper, the prospect theory, radical, neutral and benevolent cross efficiency matrices are transformed into multiplicative fuzzy hesitation preference relations. The corresponding methods are used to adjust them to satisfy acceptable consistency. Then, the deviation maximization, information entropy and Fermat-Torricelli point are used to determine the decision-maker weights, thereby aggregating the group consensus to make it satisfy the acceptable consensus degree. And sort all DMU.

From the empirical analysis results, it is found that compared with the results of traditional aggregation methods, there are differences in the ranking of the aggregation method in this paper, but no abnormalities occur. It can be used as an effective method for aggregating cross-efficiency. Among the three different aggregation methods in this paper, the results of the three methods are close, but there are also obvious differences. The result of information entropy lies between the maximization of deviation and the Fermat-Torricelli point. Therefore, decision-makers can make decisions by combining the results of the maximization of deviation and the Fermat-Torricelli point.

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