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# Cooperative Solutions in Double-Star Network Games

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## Abstract

**This paper investigates a game model known as the dual-star system, a type of multi-agent system. The study examines two-level games within such systems, where the first level represents external interactions and the second level involves internal strategic decisions. We propose a method for distributing benefits to agents in both levels. By constructing characteristic functions for the dual-star model, we establish Shapley values as the natural optimal principle for benefit allocation in both tiers of the game.**

## Keywords

**Double-star model; The Shapley value; Multi-agent system.**

## 1. Introduction

In the two-star model, all players are divided into two coalitions, each coalitions has $m$ players. each player plays a different game with his neighbors than the others. The game is divided into two levels. External games at the first level and internal games at the second level[1, 2]. Internal games within any coalition are defined as second-level games, while inter-coalition games are defined as first-level games, referred to as level-II and level-I respectively. This paper examines two cooperation models, defines characteristic functions for each alliance under these models, and employs Shapley values to allocate benefits among coalitions and players[3].

## 2. The Model

In the dual-star network G, a nonzero-sum game $\Gamma$ is defined where the network vertices are players and the edges represent connections between players. The game $\Gamma$ consists of pairwise simultaneous bimatrix games $\{\gamma_{ij}\}$ , where the games $\{\gamma_{ij}\}$ are played between neighborsi, $j \in N, i \neq j$. Let N be the set of players in the game and divide N into two coalitions $S_1$ and $S_2$, where $S_1 = \{i_1^1, \dots, i_k^1, \dots i_m^1\}$, $S_2 = \{i_1^2, \dots, i_k^2, \dots i_m^2\}$, $|S_1| \geq 2$, $|S_2| \geq 2$. $S_r(r = 1,2)$ means any coalition.

The connections of the coalitions $S_r$ are shown in Figure 1, and each coalition can cut off the connections. The player $i_m^r$ called the central player, when $i_m^r$ is connected to all the players in the coalition $S_r$ and connected to players in other coalitions, such as player $i_1^1$. In any coalition, other players can only connect to the central player.

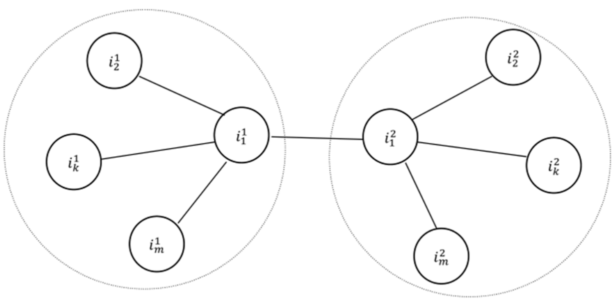


**Figure 1.** Double-star Model

Define $N_i = \{j \in N \backslash \{i\}, ij \in G\}$ as the set of neighbors of player i, where ij represents the connection between player i and j. Assuming that the player $i \in N$ and their neighbor $j \in N_i$ play a bimatrix game, the non-negative payoff matrices of plsyer i and j are $A_{ij} = [\alpha_{xy}^{ij}]_{x=1,\dots,p;y=1,\dots,l}$ and $B_{ij} = [\beta_{xy}^{ij}]_{x=1,\dots,p;y=1,\dots,l}$[4, 5]respectively.

Define $[P(i,j), L(i,j)]$ is the strategy profile of the player i and its neighbors j, where $x \in P(i，j)$ and $y \in L(i，j)$ are the strategies of the player i and j, respectively. $K_{ij}(x,y) \geq 0$ is the total payoff of player i, j in game $\gamma_{ij}$. The strategy profile of coalition $S_1$ is $P^{S_1} \cup L^{S_1} = P(i_1^1, i_2^1) \cup P(i_1^1, i_3^1) \cup \dots \cup P(i_1^1, i_m^1) \cup P(i_1^1, i_2^1) \cup L(i_1^1, i_2^1) \cup L(i_1^1, i_3^1) \cup \dots \cup L(i_1^1, i_m^1)$, The strategy profile of coalition $S_2$ is $P^{S_2} \cup L^{S_2} = P(i_1^2, i_2^2) \cup P(i_1^2, i_3^2) \cup \dots \cup P(i_1^2, i_m^2) \cup L(i_1^2, i_2^2) \cup L(i_1^2, i_3^2) \cup \dots \cup L(i_1^2, i_m^2) \cup L(i_1^2, i_1^1)$. The strategy of coalition $S_1$ and $S_2$ are respectively $\overline{X_1} \in P^{S_1} \cup L^{S_1}$、$\overline{X_2} \in P^{S_2} \cup L^{S_2}$, the payoff of coalition $S_1$ and $S_2$ are respectively $W_{S_1}(\overline{X_1}, \overline{X_2})$、$W_{S_2}(\overline{X_1}, \overline{X_2})$.

## 3. Cooperation in the Game

This paper divides the game into two levels, and defines the game within any coalition $S_r$ as the second-level game and the game between coalitions $S_1$ and $S_2$ as the first-level game, referred to as level-II and level-I. In this section, we consider the cooperation of palyer at level-II and level-I respectively, and consider two cooperation methods at level-II.

### 3.1. Cooperation at level-II

When considering the first cooperative approach in level-II, for coalition $M \subseteq S_r$, $V(M)$ represents the maxmin value of the coalition $M$ and its complement $S_r \backslash M$ engaging in a two-player zero-sum game. The function $V(M)$ is referred to as the characteristic function. Let the maxmin value of player $i(j)$ and its neighbor $j(i)$ in the game $\gamma_{ij}$ be

$$\begin{cases} \omega_{ij} = max_x min_y \alpha_{xy}^{ij}, x = 1, \dots, p; y = 1, \dots, l \\ \omega_{ji} = max_y min_x \beta_{xy}^{ij}, x = 1, \dots, p; y = 1, \dots, l \end{cases}. \tag{1}$$

The characteristic function of each coalition $M \subseteq S_r$ is,

$$V(M) = \begin{cases} \frac{1}{2}\sum_{i\in S_r}\sum_{j\in N_i \cap S_r} max_{xy}\left(\alpha_{xy}^{ij} + \beta_{xy}^{ij}\right), & M = S_r \\ \frac{1}{2}\sum_{i\in M}\sum_{j\in N_i \cap M} max_{x,y}\left(\alpha_{xy}^{ij} + \beta_{xy}^{ij}\right) + \sum_{i\in M}\sum_{k\in N_i \backslash M}\omega_{ik}, & M \subset S_r \\ \sum_{j\in N_i}\omega_{ij}, & M = \{i\} \\ 0, & M = \emptyset \end{cases}. \tag{2}$$

Assuming player $i$ and neighbor $j$j choose strategies $\bar{x}$ and $\bar{y}$ respectively to maximize their joint payoff in game $\gamma_{ij}$, where $K_{ij}(\bar{x}, \bar{y}) = max_{xy} K_{ij}(x,y)$. Consider the second cooperative method at level-II, where $V(M)$ is defined as the product of payments and $\eta(\eta \in (0,1))$ under the strategy that maximizes their joint payment, and $V(M)$ is called the characteristic function. Let the value of player $i(j)$ and its neighbor $j(i$ in the game $\gamma_{ij}$ be,

$$\begin{cases} \theta_{ij} = \alpha_{\overline{xy}}^{ij}, \\ \theta_{ji} = \beta_{\overline{xy}}^{ij}. \end{cases} \tag{3}$$

The characteristic function of each coalition $M \subseteq S_r$ is,

$$V(M)=\begin{cases}\frac{1}{2}\sum_{i\in S_r}\sum_{j\in N_i\cap S_r} max_{xy}(\alpha_{xy}^{ij}+\beta_{xy}^{ij}), & M=S_r\\ \frac{1}{2}\sum_{i\in M}\sum_{j\in N_i\cap M} max_{x,y}(\alpha_{xy}^{ij}+\beta_{xy}^{ij})+\eta\sum_{i\in M}\sum_{k\in N_i\backslash M}\theta_{ik}, & M\subset S_r\\ \eta\sum_{j\in N_i}\theta_{ij}, & M=\{i\}\\ 0, & M=\emptyset\end{cases}. \quad (4)$$

### 3.2. Cooperation at level-I

coalition $S_1$ and neighbor $S_2$ in level-I play the bimatrix game $\gamma_{S_1S_2}$. In this game, only two players $S_1$ and $S_2$ are considered. For level-I, only one cooperation method is considered, the characteristic function $V(N), N=\{S_1,S_1\}$ as shown in the formula.

$$V(N)=\frac{1}{2}\sum_{i\in M}\sum_{j\in N_i} max_{x,y}(\alpha_{xy}^{ij}+\beta_{xy}^{ij}). \quad (5)$$

## 4. The Shapley value

In this section, consider using Shapley value to assign payoff to the players at level-II and level-I respectively.

### 4.1. The Shapley value of level-I

For level-I ($N=\{S_1,S_2\}$), only one cooperation method is considered, and the Shapley value is represented by $Sh=[Sh_{S_1},Sh_{S_2}]$. For any coalition $S_r$, the Shapley value can be expressed as,

$$Sh_{S_r}=\sum_{j=i_2^r}^{i_m^r} max_{x,y}\left(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j}\right)+\frac{1}{2}max_{xy}\left(\alpha_{xy}^{i_1^1 i_1^2}+\beta_{xy}^{i_1^1 i_1^2}\right). \quad (6)$$

### 4.2. The Shapley value of level-II

For level-II, when only the coalitione $S_r$ is considered, the Shapley value of the player $i_k^r$ in the game is $Sh_{i_k^r}$ and defined the payoff of any player $i_k^r \in S_r$ is $Sh_{i_k^r}$.

In the first cooperation method, when the player $j$ is the neighbor of player $i_k^r$, the Shapley value $Sh_{i_k^r}$ of the player $i_k^r$ can be,

$$\begin{cases}Sh_{i_k^r}=\frac{1}{2}[V(i_1^r)+\sum_{j\neq i_1^r}(max_{xy}\left(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j}\right)-V(j))], k=1,\\ Sh_{i_k^r}=\frac{1}{2}\left[max_{xy}\left(\alpha_{xy}^{i_1^r i_k^r}+\beta_{xy}^{i_1^r i_k^r}\right)+V(i_k^r)-\omega_{i_1^r i_k^r}\right], k\neq 1.\end{cases} \quad (7)$$

In the second cooperation method, when the player $j$ is the neighbor of player $i_k^r$, the Shapley value $Sh_{i_k^r}$ of the player $i_k^r$ can be,

$$\begin{cases}Sh_{i_k^r}=\frac{1}{2}[V(i_1^r)+\sum_{j\neq i_1^r}(max_{xy}\left(\alpha_{xy}^{i_1^r j}+\beta_{xy}^{i_1^r j}\right)-V(j))], k=1,\\ Sh_{i_k^r}=\frac{1}{2}\left[max_{xy}\left(\alpha_{xy}^{i_1^r i_k^r}+\beta_{xy}^{i_1^r i_k^r}\right)+V(i_k^r)-\eta\theta_{i_1^r i_k^r}\right], k\neq 1.\end{cases} \quad (8)$$

## 5. Example

Consider the following case, when the $m$ players in coalition $S_1$ play a Prisoner's Dilemma game with their neighbors, for all players $i \in S_1, j \in N_i$, $A_{ij} = A, B_{ij} = B$. when the $m$ players in coalition $S_2$ play a Battle of the Sexes game with their neighbors, for all players $i \in S_2, j \in N_i \cap S_2$, $A_{ij} = C, B_{ij} = D$. (only consider the second method of cooperation)

$$A = B^T = \begin{pmatrix} b & 0 \\ a+b & a \end{pmatrix}, \quad 0 < a < b.$$

$$C = \begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}, D = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, 0 < c < d.$$

In the second cooperative method, in order to obtain the Shapley value, the characteristic function $V(M)$ of $M \subseteq S_1$ and $M \subseteq S_2$ is first determined respectively.

- $M \subseteq S_1$

$$V(M) = \begin{cases} 2b(m-1), & M = S_1 \\ 2b(|M|-1) + \eta b(m-|M|-1), & M \subset S_1, i_1^1 \in M, \\ \eta|M|b, & M \subset S_1, i_1^1 \notin M, \\ 0, & M = \emptyset. \end{cases} \quad (9)$$

- $M \subseteq S_2$

$$V(M) = \begin{cases} (d+c)(m-1), & M = S_2, \\ (d+c)(|M|-1) + \eta d(m-|M|) + \eta b, & M \subset S_2, i_1^1 \in M, \\ \eta|M|c, & M \subset S_2, i_1^1 \in M, \\ 0, & M = \emptyset. \end{cases} \quad (10)$$

The coalition $N = \{S_1, S_2\}$ characteristic function $V(N)$ is,

$$V(N) = 2b(m-1) + (d+c)(m-1) + 2b = 2bm + (d+c)(m-1). \quad (11)$$

the Shapley values $Sh_{S_1}$、$Sh_{S_2}$、$Sh_j$ of the coalition $S_1, S_2$ and players $j$ in the game are respectively as,

$$Sh_{S_1} = 2b(m-1) + \frac{2b}{2} = b(2m-1). \quad (12)$$

$$Sh_{S_2} = (d+c)(m-1) + \frac{2b}{2} = (d+c)(m-1) + b. \quad (13)$$

$$Sh_j \begin{cases} (m-1)b + \frac{1}{2}\eta b, & j = i_1^1, \\ b, & j \in S_1, j \neq i_1^1. \end{cases} \quad (14)$$

$$Sh_j \begin{cases} \frac{1}{2}[(m-1)(\eta d - \eta c + d + c) + \eta b], j = i_1^2, \\ \frac{1}{2}[(d+c) + \eta(c-d)], \qquad j \in S_2, j \neq i_1^2. \end{cases} \tag{15}$$

## 6. Summary

This study investigates unique pairwise interaction between players, employing a double-star network model to represent their connections. Building upon this framework, we develop characteristic functions and establish a simplified Shpley value specifically tailored for the double-star model, thereby streamlining the benefit-sharing process between alliances and strategic agents.

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