Study of the Relationship Between Two Types of Mixed-Order Evolution Equations and the Existence of Their Solutions

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Abstract. The purpose of this article is to study the relationship between two types of mixed-order evolution equations and the existence of their solutions. Initially, the mild solutions of the two system are obtained by Laplace transform. Then, we use semigroup theorem and sector operator theorem to get the norm estimation results of under- standing operator through special paths in the complex plane. Further, the sufficient conditions for existence and uniqueness of mild solution of the proposed system are verified by applying fixed point theorems. Finally, examples are provided to illustrate the main results.

Keywords: Mixed-order Evolution Equation; Mild Solution; Sectorial Operator.

1. Introduction

Fractional calculus is a theory about arbitrary-order differentiation and integral, and it is a generalization of integer-order calculus. In recent years, fractional order derivatives have become an important tool for describing various complex mechanical behaviors, physical behaviors and other relevant behavioral features [1–4]. As a research direction with practical significance, the fractional order differential equations have been of great concern to researchers. Many researchers have studied Riemann-Liuville fractional order differential equations and Caputo fractional order differential equations, and have achieved significant results [5–9]. For example, Arara, A et al. [8] used the fixed point theorem of Schauder combined with the diagonalization method to prove the existence of bounded solutions of a boundary value problem on an unbounded domain for differential equations involving the Caputo fractional derivative. Shu et al. [9] investigated the existence of the extremal solutions for a class of fractional partial differential equations with order $1 < \alpha < 2$ by upper and lower solution method. In this article, we study the relationship between two types of mixed-order development equations and existence of their solutions.

In addition, with the development of operator theory, people are no longer lim- ited to the study of linear fractional differential equations, and the research for a class of fractional semilinear integro-differential equation has also attracted widespread at- tention, and a lot of scholars have done relevant researches [10– 14]. Although many scholars have studied many kinds of differential equations, there are still parts waiting to be explored. Moreover, Shu etal.[12] have studied a class of fractional differential equations with nolocal conditions of order $1 < \alpha < 2$ and obtain the existence results by the fixed point theorem combined with solutions operator theorems. The system is as follows:

$$\begin{cases} D_t^{\alpha}u(t) = Au(t) + f(s,u(s)) + \int_0^t q(t-s)g(s,u(s))ds, \ t \in [0,T], \\ u(0) + m(u) = u_0 \in \mathbb{X}, \qquad u'(0) + n(u) = u_1 \in \mathbb{X}. \end{cases}$$

The system is on Banach space X and D_t^{α} is Caputo's fractional derivative of $1 < \alpha < 2$, A is a sectorial operator of type (M,θ,α,μ).

Motivated by the literature above, on a Banach space X, we study the relationship between following two types of mixed-order development equations and the existence of the solution:

$$\begin{cases} {}^L D_{0+}^{\alpha}({}^C D_{0+}^{\beta}u)(t) = Au(t) + f(t,u(t)), \quad t \in [0,T], \\ u(0) = u_0, \ I_{0+}^{1-\alpha}({}^C D_{0+}^{\beta}u)(0+) = r_0. \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}^{C}D^{\beta}_{0+}({}^{L}D^{\alpha}_{0+}u)(t) = Au(t) + f(t, u(t)), \quad t \in [0, T], \\ I^{1-\alpha}_{0+}u(0+) = u_1, \ ({}^{L}D^{\alpha}_{0+}u)(0) = r_1. \end{cases} \tag{1.2}$$

where $t \in J = [0, \infty)$, $0 < \alpha < 1$, $0 < \beta < 1$, $1 < \alpha+\beta < 2$. Here D^{α}_{0+} denotes Riemann-Liouville fractional order derivative of order α with lower limit zero(see definition 2.2),and ${}^{C}D^{\beta}_{0+}$ denotes Caputo fractonal order derivative of order β lower limit zero(see definition 2.3); Let $A : D(A) \subseteq X \to X$ be a sectorial operator of type $(M,\theta,\alpha+\beta,\mu)$(see definition 2.4), and the nonlinear map $f : J \times X \to X$ is a continious function satisfying some conditions given later.

The motives and highlights in this paper are as follows:

For equation (1) and (2), instead of just using Riemann-Liouville fractional derivative or Caputo fractional derivative, it combines two differentials to construct a new class of equations. Meanwhile, We obtain the solution of equation (1.1) and (1.2) not by the traditional integral operator sequentially acting on the equation, but by using the properties of the Laplace operator and the sectorial operator to obtain the mild solution of the equation, which is different from the research methods of previous researchers.

2. Preliminaries

In this section, we will present some primary components, including notations, definitions, lemmas, theorems, and so on, which are required in the process to prove our main results. In this paper, $C(J, X)$ is the Banach space of all continuous functions from $J = [0, \infty)$ into X, furnished with the uniform convergence norm $//u//_{\infty}$, and denote $C_b(J, X) = \{f \in C(J, X) : f \text{ is bounded.}\}$, endowed with the norm of uniformly convergence as well.

Definition 2.1 [5](Riemann-Liouville Fractional Integral)

The Riemann-Liouville fractional integral of order $\alpha \in R_+$ afunction $u \in L^1([0, \infty);R^+)$ of oreder $\alpha \in R_+$ is defined by

$$I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}u(s)ds,$$

where Γ(.) is the Gamma function.

Definition 2.2 [5](Riemann-Liouville fractional order derivative)

The Riemann-Liouville fractional order derivative of order $\alpha \in R_+$ of a function f given on the interval $[0, \infty)$ is defined by

$${}^{L}D^{\alpha}_{0+}u(t) = (\tfrac{d}{dx})^n (I^{n-\alpha}_{0+}u)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $\alpha \in (n-1, n), n \in N$.

Definition 2.3 [5](Caputo fractonal order derivative)

The Caputo fractional order derivative of order $\alpha \in R_+$ of a function u given on the interval $[0, \infty)$ is defined by

$${}^{C}D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds = I^{n-\alpha}_{0+}u^{(n)}(t),$$

where $\alpha \in (n-1, n), n \in N$.

Definition 2.4 [12] $A : D(A) \subset X \to X$ be a closed linear operator.A is said to be sectorial operator of type (M,θ,α,μ) if there exist $0 < \theta < \frac{\pi}{2}$, $M > 0$ *and* $\mu \in \mathbb{R}$ *such* that the α-resolvent of A exsits outside the sector

$$\mu + S_{\theta} = \{\mu + \lambda^{\alpha} : \lambda \in C, |Arg(-\lambda^{\alpha})| < \theta\},$$

and let $R(\lambda^{\alpha}, A) = (\lambda^{\alpha} I - A)^{-1}$, the following relationship is established $\|R(\lambda^{\alpha}, A)\| \leq \frac{M}{|\lambda^{\alpha}-\mu|}$, $\lambda^{\alpha} \notin \mu + S_{\theta}$.

Lemma 2.1 [7] If $Re(\alpha) > 0$, $n = [Re(\alpha)] + 1$, $u(t) \in AC^n[0, b]$, $\forall b > 0$ and $|u(t)| \leq Be^{q_0 t}(t > b > 0)$ for $B > 0$, $q_0 > 0$, and there exists the finite limits

$$\lim_{t\to 0^+}\left[D^k I_{0+}^{n-\alpha}u(t)\right]$$

and

$$\lim_{t\to\infty}\left[D^k I_{0+}^{n-\alpha}u(t)\right]=0.$$

then

$$(\mathcal{L}^L D_{0+}^{\alpha}u)(\lambda)=\lambda^{\alpha}(\mathcal{L}u)(\lambda)-\sum_{k=0}^{n-1}\lambda^{n-k-1}D^k(I_{0+}^{n-\alpha}u(0+))(Re(\lambda)>q_0).$$

Lemma 2.2 [7] Let $\alpha>0, n-1<\alpha\le n(n\in N)$ be such that $u(t)\in C^n(R_+)$, $u^{(n)}(t)\in L_1(0,b)$, $\forall b>0$, and $|u(t)|\le Be^{q_0 t}(t>b>0)$ for $B>0$, $q_0>0$,the Laplace transforms $(Lu)(\lambda)$ and $(LD^n u)(\lambda)$ exist, and

$$\lim_{t\to+\infty}(D^k u)(t)=0, for\ k=0,1,\ldots,n-1.$$

Then the following relation holds:

$$(\mathcal{L}^C D_{0+}^{\alpha}u)(\lambda)=\lambda^{\alpha}(\mathcal{L}u)(\lambda)-\sum_{k=0}^{n-1}\lambda^{\alpha-k-1}(D^k u)(0)$$

Theorem 2.1 [15] (Banach's fixed point theorem)

Let X be a nonempty complete metric space, $T: X\to X$ is a compression map, then T must have a unique fixed point.

In the following pages, what we hope is to be able to study the relationship and the existence of the mild solutions of the two types of mixed-order fractional differential equation (1.1) and (1.2). We first consider the definition of mild solutions to system (1.1) and system (2.2), and Operator estimation is made on the solution operator of the mild solutions. Then, we define an operator G according to the mild solutions obtained, and the main existence results can be acquired by applying the fixed-point theorem.

3. Definition of a Mild Solution to the Mixed-order Fractional Evolution Equation (1.1)

Firstly, we consider the following cauchy problem

$$\begin{cases} {}^L D_{0+}^{\alpha}({}^C D_{0+}^{\beta}u)(t)=Au(t)+f(t), & t\in[0,T],\\ u(0)=u_0,\ I_{0+}^{1-\alpha}({}^C D_{0+}^{\beta}u)(0+)=r_0. \end{cases} \tag{3.1}$$

where A is a sectorial operator of type $(M, \theta, \alpha+\beta, \mu)$.

Theorem 3.1 The function f satisfies the consistent holder condition, then the unique solution of cauchy problem (3.1) is given by

$$u(t)=S_{\alpha+\beta}(t)u_0+T_{\alpha+\beta}(t)r_0+\int_0^t T_{\alpha+\beta}(t-s)f(s)ds, \tag{3.2}$$

where

$$T_{\alpha+\beta}(t)=\frac{1}{2\pi i}\int_c e^{\lambda t}R(\lambda^{\alpha+\beta},A)d\lambda,$$

$$S_{\alpha+\beta}(t)=\frac{1}{2\pi i}\int_c e^{\lambda t}\lambda^{\alpha+\beta-1}R(\lambda^{\alpha+\beta},A)d\lambda,$$

with c being a suitable path such that $\lambda^{\alpha+\beta}\notin\mu+S_\theta$ for $\lambda^{\alpha+\beta}\in c$.

Remark 1 we can observe that A is the infinitesimal generator of a $\alpha+\beta$-reslovent family $\{T_{\alpha+\beta}(t)\}_{t\geq 0}$ and $\{S_{\alpha+\beta}(t)\}_{t\geq 0}$ in Banamch space, and $T_{\alpha+\beta}(t)$ and $S_{\alpha+\beta}(t)$ are well definitions.

Proof. We perform the Laplace transform on the left side of the equation

$$\mathcal{L}\left[{}^{L}D^{\alpha}_{0+}({}^{C}D^{\beta}_{0+}u)(t)\right](\lambda) = \lambda^{\alpha}\mathcal{L}({}^{C}D^{\beta}_{0+}u(t))(\lambda) - I^{1-\alpha}_{0+}({}^{C}D^{\beta}_{0+}u)(0+)$$
$$= \lambda^{\alpha}(\lambda^{\beta}\mathcal{L}u(\lambda) - \lambda^{\beta-1}u(0)) - I^{1-\alpha}_{0+}({}^{C}D^{\beta}_{0+}u)(0+)$$
$$= \lambda^{\alpha+\beta}Lu - \lambda^{\alpha+\beta-1}u(0) - I^{1-\alpha}_{0+}({}^{C}D^{\beta}_{0+}u)(t)$$

It follows that

$$\lambda^{\alpha+\beta}(Lu)(\lambda) - \lambda^{\alpha+\beta-1}u(0) - I^{1-\alpha}_{0+}({}^{C}D^{\beta}_{0+}u)(t) = L[Au(t) + f(t)](\lambda)$$

$$(\lambda^{\alpha+\beta}I - A)(Lu)(\lambda) = \lambda^{\alpha+\beta-1}u_0 + r_0 + (Lf)(\lambda)$$

let

$$R(\lambda^{\alpha+\beta}, A) = (\lambda^{\alpha+\beta}I - A)^{-1}$$

$$(Lu)(\lambda) = R(\lambda^{\alpha+\beta}, A)(\lambda^{\alpha+\beta-1}u_0 + r_0 + (Lf)(\lambda))$$

$$u(t) = L^{-1}R(\lambda^{\alpha+\beta}, A)(\lambda^{\alpha+\beta-1}u_0 + r_0) + L^{-1}R(\lambda^{\alpha+\beta}, A)Lf$$

Now we can get (3.2) easily by converting the above equation.

Theorem 3.2 The function f satisfies the consistent holder condition, then the solu- tions of the cauchy problem (1.1) are fixed points of operator equation

$$u(t) = S_{\alpha+\beta}(t)u_0 + T_{\alpha+\beta}(t)r_0 + \int_0^t T_{\alpha+\beta}(t\text{-}s)f(s,u(s))ds,$$

Theorem 3.2 leads the following appropriate definition of a mild solution to (1.1).

Definition 3.1 Afunction $u \in C(J,X)$ is called a mild solution of (1.1) when it satifies the operator equation

$$u(t) = S_{\alpha+\beta}(t)u_0 + T_{\alpha+\beta}(t)r_0 + \int_0^t T_{\alpha+\beta}(t-s)f(s,u(s))ds, \tag{3.3}$$

4. Definition of a Mild Solution to the Mixed-order Fractional Evolution Equation (1.2)

Similarly, we consider the following cauchy problem

$$\begin{cases} {}^{C}D^{\beta}_{0+}({}^{L}D^{\alpha}_{0+}u)(t) = Au(t) + f(t), & t \in [0,T], \\ I^{1-\alpha}_{0+}u(0+) = u_1, \ ({}^{L}D^{\alpha}_{0+}u)(0) = r_1. \end{cases} \tag{4.1}$$

where A is a sectorial operator of type $(M,\theta,\alpha+\beta,\mu)$.

Theorem 4.1 The function f satisfies the consistent holder condition, then the unique solution of cauchy problem (4.1) is given by

$$u(t) = S_{\alpha+\beta,\beta}(t)u_1 + S_{\alpha+\beta,\beta-1}(t)r_1 + \int_0^t T_{\alpha+\beta}(t-s)f(s)ds. \tag{4.2}$$

where

$$S_{\alpha+\beta,\beta}(t) = \frac{1}{2\pi i}\int_c e^{\lambda t}\lambda^{\beta}R(\lambda^{\alpha+\beta}, A)d\lambda,$$

$$S_{\alpha+\beta,\beta-1}(t) = \frac{1}{2\pi i}\int_c e^{\lambda t}\lambda^{\beta-1}R(\lambda^{\alpha+\beta}, A)d\lambda$$

$$T_{\alpha+\beta}(t) = \frac{1}{2\pi i}\int_c e^{\lambda t} R(\lambda^{\alpha+\beta}, A)d\lambda,$$

with c being a suitable path such that $\lambda^{\alpha+\beta} \notin \mu + S_\theta$, for $\lambda^{\alpha+\beta} \in c$.

Remark 2 Similarly, we can also see that A is the infinitesimal generator of a $\alpha+\beta$- reslovent family $\{S_{\alpha+\beta,\beta}(t)\}_{t\geq 0}$, $\{S_{\alpha+\beta,\beta-1}(t)\}_{t\geq 0}$ and $\{T_{\alpha+\beta}(t)\}_{t\geq 0}$ in Banach space, and $S_{\alpha+\beta,\beta}(t)$,$S_{\alpha+\beta,\beta-1}(t)$ and $T_{\alpha+\beta}(t)$ are well definitions.

Proof. We perform the Laplace transform on the left side of the equation

$$\begin{aligned}\mathcal{L}\left[{}^C D^\alpha_{0+}({}^L D^\beta_{0+}u)(t)\right](\lambda) &= \lambda^\beta \mathcal{L}({}^L D^\alpha_{0+}u(t))(\lambda) - \lambda^{\beta-1}({}^L D^\alpha_{0+}u)(0)\\ &= \lambda^\beta(\lambda^\alpha \mathcal{L}u(\lambda) - I^{1-\alpha}_{0+}u(0+)) - \lambda^{\beta-1}({}^L D^\alpha_{0+}u)(0)\\ &= \lambda^{\alpha+\beta}\mathcal{L}u - \lambda^{\beta-1}I^{1-\alpha}_{0+}u(0+) - \lambda^{\beta-1}({}^L D^\alpha_{0+}u)(0)\end{aligned}$$

It follows that

$$\lambda^{\alpha+\beta}Lu - \lambda^{\beta-1}I^{1-\alpha}_{0+}u(0+) - \lambda^{\beta-1}({}^L D^\alpha_{0+}u)(0) = L[Au(t) + f(t)](\lambda)$$

$$(\lambda^{\alpha+\beta}I - A)(Lu)(\lambda) = \lambda^\beta u_1 + \lambda^{\beta-1}r_1 + (Lf)(\lambda)$$

let

$$R(\lambda^{\alpha+\beta}, A) = (\lambda^{\alpha+\beta}I - A)^{-1}$$

$$(Lu)(\lambda) = R(\lambda^{\alpha+\beta}, A)(\lambda^\beta u_1 + \lambda^{\beta-1}r_1 + (Lf)(\lambda))$$

$$u(t) = L^{-1}R(\lambda^{\alpha+\beta}, A)(\lambda^\beta u_1 + \lambda^{\beta-1}r_1) + L^{-1}R(\lambda^{\alpha+\beta}, A)Lf$$

Now we can get (3.1) easily by converting the above equation.

Theorem 4.2 The function f satisfies the consistent Holder condition and A is a sec- torial operator of type $(M,\theta,\alpha+\beta,\mu)$, then the solutions of the cauchy problem (1.2) are fixed points of operator equation

$$u(t) = S_{\alpha+\beta,\beta}(t)u_1 + S_{\alpha+\beta,\beta-1}(t)r_1 + \int_0^t T_{\alpha+\beta}(t-s)f(s,u(s))ds. \tag{4.3}$$

Theorem 4.2 leads the following appropriate definition of a mild solution to (1.2).

Definition 4.1 Afunction $u \in C(J,X)$ is called a mild solution of (1.2) when it satifies the operator equation

$$u(t) = S_{\alpha+\beta,\beta}(t)u_1 + S_{\alpha+\beta,\beta-1}(t)r_1 + \int_0^t T_{\alpha+\beta}(t-s)f(s,u(s))ds. \tag{4.4}$$

By observing the expression forms of the solutions of these two types of mixed frac- tional order evolution equations, the solution operator forms of these mixed fractional equations are different after the differential order is exchanged. Therefore, we perform a norm estimation of its solution operator.

5. Norm Estimations

Theorem 5.1 Let A be an operator with type $(M,\theta,\alpha+\beta,\mu)$. Then we can show the following estimates on $\|S_{\alpha+\beta,\beta}(t)\|$.

(i) When $\mu \geq 0$, for $\phi \in (0,\pi)$, we have

$$\|S_{\alpha+\beta,\beta}(t)\| \leq \frac{K_1(\theta,\phi)^{\frac{1}{\alpha}} M e^{[K_1(\theta,\phi)(\mu t^{\alpha+\beta}+1)^{\frac{1}{\alpha+\beta}}]}[(1+\frac{\sin\phi}{\sin(\phi-\theta)})^{\frac{1}{\alpha+\beta}}-1]t^{\alpha+\beta}(\mu+\frac{1}{t^{\alpha+\beta}})^{\frac{1}{\alpha}}(\mu+\frac{1}{t^{\alpha+\beta}})^{\frac{1}{\alpha+\beta}}}{\pi\sin\theta}$$
$$+\frac{\Gamma(\beta+1)Mt^{\alpha-1}}{\pi(1+\mu t^{\alpha+\beta})|\cos(\frac{\pi-\phi}{\alpha+\beta})|^{\beta+1}\sin\theta\sin\phi}.$$

(ii) When $\mu < 0$. For $\varphi \in (0,\pi)$, we have

$$\|S_{\alpha+\beta,\beta}(t)\| \leq \frac{t^{\alpha-1}eM[(1+\sin\phi)^{\frac{1}{\alpha+\beta}}-1]}{\pi(1+|\mu|t^{\alpha+\beta})|\cos\phi|} + \frac{\Gamma(\beta+1)Mt^{\alpha-1}}{\pi(1+|\mu|t^{\alpha+\beta})|\cos\frac{\pi-\phi}{\alpha+\beta}|^{\beta+1}|\cos\phi|}.$$

For $t>0$, *where* $K_1(\theta,\phi)=\max\{1,\frac{\sin\theta}{\sin(\phi-\theta)}\}$.

Proof. Give a $\phi \in (0,\pi)$, define $S_\phi = \{\lambda \in C : |Arg(-\lambda)| < \phi\}$. First, we prove (i). For $t > 0$, consider the positively oriented path c which is the image of the boundary of $(\mu + \frac{1}{t^{\alpha+\beta}} + S_\theta) \cup S_\phi$ under the function $P(\frac{1}{z^{\alpha+\beta}})$, here we require $\phi > \theta$ so that $S_\phi \not\subset (\mu + \frac{1}{t^{\alpha+\beta}} + S_\theta)$(as show in Fig.1). Then along c, the resolvent $(\lambda^{\alpha+\beta}I - A)$ is well defined, and then the representation of $S_{\alpha+\beta,\beta}(t)$ is meaningful. Now, we let c_1 be the part of c associated with the part on the boundary from z_t to z_t, and c_{21} and c_{22} be the parts of c associated with the parts on the boundary from infinity to z_t and from z_t to infinity, where z_t and z_t are the intersection points of the boundaries of $\mu + \frac{1}{t^{\alpha+\beta}} + S_\theta$ and S_ϕ.

In this case, we divide $S_{\alpha+\beta,\beta}(t)$ into two parts: $S_{\alpha+\beta,\beta}(t) = I_1(t) + I_2(t)$, here

—
$$I_1(t) = \frac{1}{2\pi i}\int_c \lambda^\beta e^{\lambda t}(\lambda^{\alpha+\beta}, A)^{-1}d\lambda$$

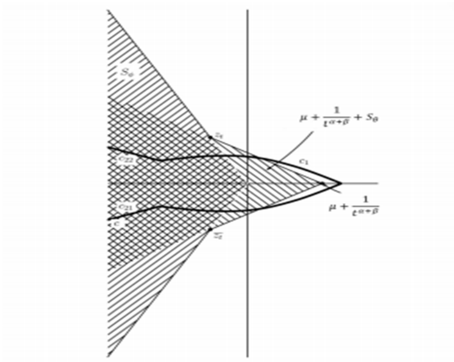

Fig 1. A particular path for estimating $//S_{\alpha+\beta,\beta}(t)//$ when $\mu \geq 0$

and

$$I_2(t) = \frac{1}{2\pi i}\int_{c_{21}\cup c_{22}} \lambda^\beta e^{\lambda t}(\lambda^{\alpha+\beta}, A)^{-1}d\lambda.$$

For $\lambda \in c_1$, one can easily see from the Fig.1 that

$$(\mu + \frac{1}{t^{\alpha+\beta}})\sin\theta \leq |\lambda|^{\alpha+\beta} \leq K_1(\theta,\phi)(\mu + \frac{1}{t^{\alpha+\beta}}),$$

And $\frac{1}{|\lambda^{\alpha+\beta}-\mu}| \leq \frac{t^{\alpha+\beta}}{\sin\theta}$.

It is obvious that c_1 is symmetric about the real axis and the part above the real axis is parameterized

$$f(v) = (\mu + \frac{1}{t^{\alpha+\beta}} - v\cos\theta + iv\sin\theta)^{\frac{1}{\alpha+\beta}}, 0 \leq v \leq \frac{\sin\phi}{\sin(\theta-\phi)(\mu+\frac{1}{t^{\alpha+\beta}})} := l,$$

then

$$l(c_1) = 2\int_0^l |f'(v)|dv = \frac{2}{\alpha+\beta}\int_0^l [(\mu + \frac{1}{t^{\alpha+\beta}}) + |v|]^{\frac{1}{\alpha+\beta}-1}dv$$
$$= 2[(1 + \frac{\sin\theta}{\sin(\phi-\theta)})^{\frac{1}{\alpha+\beta}} - 1](\mu + \frac{1}{t^{\alpha+\beta}})^{\frac{1}{\alpha+\beta}}$$

Therefore,

$$\|I_1(t)\| \leq \frac{1}{2\pi}\int_{c1} |e^{\lambda t}||\lambda^{\alpha-r}|\|(\lambda^\alpha - A)^{-1}\||d\lambda|$$
$$\leq \frac{K_1(\theta,\phi)Me^{K_1(\theta,\phi)(\mu t^{\alpha+\beta}+1)^{\frac{1}{\alpha+\beta}}}[(1+\frac{\sin\phi}{\sin(\phi-\theta)})^{\frac{1}{\alpha+\beta}}-1]t^{\alpha+\beta}(\mu+\frac{1}{t^{\alpha+\beta}})^{\frac{1}{\alpha}}(\mu+\frac{1}{t^{\alpha+\beta}})^{\frac{1}{\alpha+\beta}}}{\pi \sin\theta}.$$

Now we come to estimate $//I_2(t)//$. Note that for $t > 0$,

$$|z_t| = |\overline{z_t}| \geq (\mu + \frac{1}{t^{\alpha+\beta}}) \sin\theta, \quad t > 0$$

and for $\lambda \in c_2 = c_{21} \cup c_{22}$,

$$|\lambda^{\alpha+\beta} - \mu| \geq |z_t|\sin\phi = |\overline{z_t}|\sin\phi$$

$$\|I_2(t)\| \leq \frac{1}{\pi}\int_{c_{22}} |\lambda|^\beta |e^{t\lambda}| \|(\lambda^{\alpha+\beta}I - A)^{-1}\||d\lambda|$$
$$\leq \frac{M}{\pi}\int_{c_{22}} |e^{t\lambda}| \frac{|\lambda|^\beta}{|\lambda^{\alpha+\beta}-\mu|}|d\lambda|$$
$$\leq \frac{Mt^{\alpha+\beta}}{\pi(1+\mu t^{\alpha+\beta})\sin\theta\sin\phi}\int_{c_{22}} e^{Re(t\lambda)}|\lambda|^\beta|d\lambda|$$
$$\leq \frac{Mt^{\alpha+\beta}}{\pi(1+\mu t^{\alpha+\beta})\sin\theta\sin\phi}\int_0^\infty e^{-t\tau|\cos(\frac{\pi-\phi}{\alpha+\beta})|}\tau^\beta d\tau$$
$$= \frac{\Gamma(\beta-1)Mt^{\alpha-1}}{\pi(1+\mu t^{\alpha+\beta})|\cos(\frac{\pi-\phi}{\alpha+\beta})|^2\sin\theta\sin\phi}.$$

In the above, we require $\cos\frac{\pi-\phi}{\alpha+\beta} < 0$ or equivalently $\frac{\pi-\phi}{\alpha} \in (\frac{\pi}{2},\pi)$. Therefore, by combing the estimates of $//I_1(t)//$ and $//I_2(t)//$, we get the conclusion.

Now we turn to prove (ii). The porove is similar to that of (i). In this case, we consider the path of c, whose image under the $P(z^{\alpha+\beta})$ is the boundary of $(\frac{1}{t^{\alpha+\beta}} + S_{\phi-\frac{\pi}{2}}) \cup S_\phi$, here we require $\phi \in (\frac{\pi}{2},\pi)$(as show in Fig 2). As same as the before, let c_1 be the path of c_1 be the part of c associated with the part on the boundary from z_t to z_t, and c_{21} and c_{22} respectively represent the parts of c associated with the parts on the boundary from infinity to z_t and from z_t to infinity, where z_t and z_t are the intersection points of the boundaries of $\frac{1}{t^{\alpha+\beta}} + S_{\phi-\frac{\pi}{2}}$ and S_ϕ.

For $\lambda \in c_1$, combine with the Fig.2 above, we can get easily see that

$$\frac{1}{|\lambda^{\alpha+\beta}-\mu|} \leq \frac{t^{\alpha+\beta}}{(1+|\mu|t^{\alpha+\beta})|\cos\phi|},$$

and

$$\frac{\cos\phi}{t^{\alpha+\beta}} \leq |\lambda|^{\alpha+\beta} \leq \frac{1}{t^{\alpha+\beta}}.$$

Note that c1 is the symmetric about the real axis and the part above the real has the parametrization, let

$$f(v) = (\frac{1}{t^{\alpha+\beta}} - v\cos\theta + iv\sin\theta)^{\frac{1}{\alpha+\beta}},$$
$$0 \leq v \leq \frac{1}{t^{\alpha+\beta}}\sin\phi := l,$$

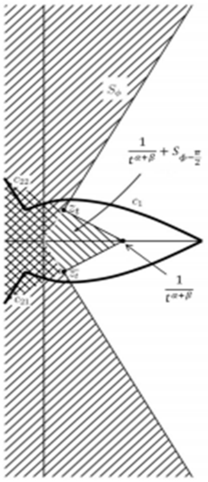

Fig 2. A particular path for estimating $//S_{\alpha+\beta,\ \beta}(t)\ //$ when < 0

then

$$\begin{aligned} l(c_1) &= 2\int_0^l |f'(v)|dv \\ &\le \frac{2}{\ \ +\ \ }\int_0^l [\frac{1}{t^{\alpha+\beta}} + |v|]^{\frac{1}{\alpha+\beta}-1}dv \\ &\le 2[(1+\sin\ \ )^{\frac{1}{\alpha+\beta}} - 1](\frac{1}{t^{\alpha+\beta}})^{\frac{1}{\alpha+\beta}} \\ &\le \frac{2}{t}[(1+\sin\ \ )^{\frac{1}{\alpha+\beta}} - 1]. \end{aligned}$$

It follows that

$$\begin{aligned} \|I_1(t)\| &\le \frac{1}{2}\int_{c_1} |e^{\lambda t}||\ \ ^{\beta}|\|(\ \ ^{\alpha+\beta}I - A)^{-1}\||d\ \ | \\ &\le \frac{1}{2}\int_{c_1} \frac{e^{t|\lambda|}|\ \ |^{\beta}M}{|\ \ ^{\alpha+\beta} -\ \ |}|d\ \ | \\ &\le \frac{t^{\alpha-1}eM[(1+\sin\ \ )^{\frac{1}{\alpha+\beta}} - 1]}{(1+|\ \ |t^{\alpha+\beta})|\cos\ \ |}. \end{aligned}$$

On the other hand, for $\in c_2$, we have

$$|\ \ ^{\alpha+\beta} -\ \ | \ge (|\ \ | + \frac{1}{t^{\alpha+\beta}})\cos\ \ .$$

Then, we have

$$\begin{aligned} \|I_2(t)\| &\le \frac{M}{\pi}\int_{c_{22}} |\lambda^{\beta}||e^{\lambda t}|\frac{M}{|\lambda^{\alpha+\beta} - \mu|}|d\lambda| \\ &\le \frac{Mt^{\alpha+\beta}}{\pi(1+\mu|t^{\alpha+\beta})|\cos\phi|}\int_{c_{22}} e^{Re(t\lambda)}|\lambda|^{\beta}|d\lambda| \\ &\le \frac{Mt^{\alpha}}{\pi(1+|\mu|t^{\alpha+\beta})|\cos\phi|}\int_0^{\infty} e^{-t\tau|\cos\frac{\pi-\phi}{\alpha+\beta}|}\tau^{\beta}d\tau, \qquad let\ \tau = \frac{s}{t|\cos(\frac{\pi-\phi}{\alpha+\beta})|} \end{aligned}$$

$$\leq \frac{Mt^{\alpha-1}}{\pi(1+|\mu|t^{\alpha+\beta})|\cos\frac{\pi-\phi}{\alpha+\beta}|^{\beta+1}|\cos\phi|}\int_0^\infty e^{-s}s^\beta ds$$
$$= \frac{\Gamma(\beta+1)Mt^{\alpha-1}}{\pi(1+|\mu|t^{\alpha+\beta})|\cos\frac{\pi-\phi}{\alpha+\beta}|^{\beta+1}|\cos\phi|}.$$

Therefore, we complete the proof of conclusion (ii).

Similarly, we can prove the following estimates on $//S_{\alpha+\beta,\beta-1}(t)//$ and $//T_{\alpha+\beta}(t)//$.

Theorem 5.2 Let A be an operator with type $(M,\theta,\alpha+\beta,\mu)$. Then we can show the following estimates on $//S_{\alpha+\beta,\beta-1}(t)//$ and $//T_{\alpha+\beta}(t)//$.

(i) When $\mu \geq 0$, for $\phi \in (0,\pi)$, we have

$$\|S_{\alpha+\beta}(t)\| \leq \frac{K_1(\theta,\phi)Me^{[K_1(\theta,\phi)(1+\mu t^{\alpha+\beta})^{\frac{1}{\alpha+\beta}}]}[(1+\frac{\sin\phi}{\sin(\phi-\theta)})^{\frac{1}{\alpha+\beta}}-1]}{\pi\sin^{1+\frac{1}{\alpha+\beta}}\theta}(1+\mu t^{\alpha+\beta})$$
$$+\frac{\Gamma(\alpha)M}{\pi(1+\mu t^{\alpha+\beta})|\cos\frac{\pi-\phi}{\alpha}|^{\alpha+\beta}\sin\theta\sin\phi},$$

$$\|S_{\alpha+\beta,\beta-1}(t)\| \leq \frac{K_1(\theta,\phi)^{\frac{1}{\alpha+1}}Me^{[K_1(\theta,\phi)(\mu t^{\alpha+\beta}+1)^{\frac{1}{\alpha+\beta}}]}[(1+\frac{\sin\phi}{\sin(\phi-\theta)})^{\frac{1}{\alpha+\beta}}-1](\mu+\frac{1}{t^{\alpha+\beta}})^{\frac{1}{\alpha+1}}(1+\mu t^{\alpha+\beta})^{\frac{1}{\alpha+\beta}}}{\pi\sin\theta}$$
$$+\frac{\Gamma(\beta)Mt^\alpha}{\pi(1+\mu t^{\alpha+\beta})|\cos(\frac{\pi-\phi}{\alpha+\beta})|^\beta\sin\theta\sin\phi},$$

and

$$\|T_{\alpha+\beta}\| \leq \frac{M[(1+\frac{\sin\phi}{\sin(\phi-\theta)})^{\frac{1}{\alpha+\beta}}]}{\pi\sin\theta}(1+\mu t^{\alpha+\beta})^{\frac{1}{\alpha+\beta}}t^{\alpha+\beta-1}e^{[K_1(\theta,\phi)(1+\mu t^{\alpha+\beta})]^{\frac{1}{\alpha+\beta}}}$$
$$+\frac{Mt^{\alpha+\beta-1}}{\pi(1+\mu t^{\alpha+\beta})|\cos\frac{\pi-\phi}{\alpha+\beta}|^{\alpha+\beta}\sin\theta\sin\phi}.$$

(ii) When $\mu < 0$. For $\varphi \in (0,\pi)$, we have

$$\|S_{\alpha+\beta}(t)\| \leq \left(\frac{eM[(1+\sin\phi)^{\frac{1}{\alpha+\beta}}-1]}{\pi|\cos\phi|^{1+\frac{1}{\alpha+\beta}}}+\frac{\Gamma(\alpha+\beta)M}{\pi|\cos\phi||\cos\frac{\pi-\phi}{\alpha+\beta}|^{\alpha+\beta}}\right)\frac{1}{1+|\mu|t^{\alpha+\beta}},$$

$$\|S_{\alpha+\beta,\beta-1}(t)\| \leq \frac{t^\alpha eM[(1+\sin\phi)^{\frac{1}{\alpha+\beta}}-1]}{\pi(1+|\mu|t^{\alpha+\beta})|\cos\phi|}+\frac{\Gamma(\beta+1)Mt^\alpha}{\pi(1+|\mu|t^{\alpha+\beta})|\cos\frac{\pi-\phi}{\alpha+\beta}|^\beta|\cos\phi|},$$

And

$$\|T_{\alpha+\beta}(t)\| \leq \frac{eMt^{\alpha+\beta-1}[(1+\sin\phi)^{\frac{1}{\alpha+\beta}}-1]}{\pi|\cos\phi|(1+|\mu|t^{\alpha+\beta})}+\frac{Mt^{\alpha+\beta-1}}{\pi|\cos\phi||\cos\frac{\pi-\phi}{\alpha+\beta}|(1+|\mu|t^{\alpha+\beta})}.$$

Fort > 0, where $K_1(\theta,\phi) = \max\{1, \frac{\sin\theta}{\sin(\phi-\theta)}\}$.

6. Existence of Mild Solutions

Next, we present existence and uniqueness result for system (1.1) and system (1.2) based on Banach fixed theorem. In order to prove the desired results about the fractional equation in this paper, from the estimates on $//S_{\alpha+\beta}(t)//$, $//S_{\alpha+\beta,\beta}(t)//$, $//S_{\alpha+\beta,\beta-1}(t)//$ and $|T_{\alpha+\beta}(t)//$, we make the following assumptions:

(H_1): The operators $S_{\alpha+\beta}(t)$, $S_{\alpha+\beta,\beta}(t)$, $S_{\alpha+\beta,\beta-1}(t)$, and $T_{\alpha+\beta}(t)$ generated by A are compact in D(A) when $t \geq 0$ and

$\sup_{t\in J} // S_{\alpha+\beta}(t) // \le M_1$, $\sup_{t\in J} // S_{\alpha+\beta,\beta}(t) // \le M_1$, $\sup_{t\in J} // S_{\alpha+\beta,\beta-1}(t) // \le M_1$, $\sup_{t\in J} // T_{\alpha+\beta}(t) // \le M_1$

(H_2): There exists a continuous function L(t) making the continuous function $f : J \times X \to X$ satisfy the Lipschitz condition:

$$//f(t,u_1(t)) - f(t,u_2(t)) // \le L(t) //u_1(t) - u_2(t) //, t \in J, u_1(t), u_2(t) \in X.$$

(H_3): L(t) and f(s,o) are continuous functions, so we can assume that there exist constants L and F satisfying $//L(t) // \le L$ and $//f(s,0) // \le F$ for $t \in [0,T]$, and $LT < 1$.

Theorem 6.1 Assume that (H_1), (H_2), (H_3) hold, then the system (1.1) has a unique mild solution u on the interval J.

Proof. Define a function $G : C_b(J,X) \to C_b(J,X)$ by

$$(Gu)(t) = S_{\alpha+\beta}(t)u_0 + T_{\alpha+\beta}(t)r_0 + \int_0^t T_{\alpha+\beta}(t-s)f(s,u(s))ds.$$

Firstly, we show that if $u \in C_b(J,X)$, then $G(u) \in C_b(J,X)$.

$$\|\mathbb{G}u(t)\| \le \|S_{\alpha+\beta}(t)u_0\| + \|T_{\alpha+\beta}(t)r_0\| + \|\int_0^t T_{\alpha+\beta}(t-s)f(s,u(s))ds\|$$
$$\le M_1|u_0| + M_1|r_0| + M_1\int_0^t \|f(s,u(s)) - f(s,0)\| + \|f(s,0)\|ds$$
$$\le M_1(|u_o| + |r_0| + LT //u // + FT).$$

Obviously, it means $G: C_b(J,X) \to C_b(J,X)$. Then, we need to prove that G is a compression mapping in G. For $u_1, u_2 \in C$, we have

$$//Gu_1(t) - Gu_2(t) // = /\int_0^t T_{\alpha+\beta}(t-s)[f(s,u_1(s)) - f(s,u_2(s))]ds //$$
$$\le LT //u_1 - u_2 //$$

According to Bananch's fixed point theorem, we know there is a fixed point in $C_b(J,X)$, which is the mild solution of system (1.1)

Similarly, we can get the following conclusion:

Theorem 6.2 Assume that (H_1), (H_2), (H_3) hold, then the system (1.2) has a unique mild solution u on the interval J.

7. Example

Example 1. As an application of our obtained results, suppose that $\Omega \subset R^2$ is a unit circular domain with respect to the origin. Consider the following fractional partial differential equation:

$$\begin{cases} {}^L D_{0+}^{\alpha}({}^C D_{0+}^{\beta}u)(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2} - u(t,x) + \frac{\cos u(t,x)}{2+t}, \ 1 < \alpha+\beta < 2, \ t \in [0,T], \ x \in \Omega, \\ u(0,x) = u_0(x), \ I_{0+}^{1-\alpha}({}^C D_{0+}^{\beta}u)(0,x) = r_0(x) \\ u(t,x) = 0, \ t \in [0,T], \ x \in \partial\Omega. \end{cases}$$

Let $X = L^2(\Omega)$, $0 < \alpha \le 1, 0 < \beta < 1$, $1 < \alpha+\beta < 2$, Define the operator

A: $D(A) \subseteq X \to X$ by $Au = \frac{\partial^2 u(t,x)}{\partial x^2} - u(t,x)$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then A is a sectorial operator of type $(M;\theta;\alpha+\beta;\mu)$ with $\mu = -1$, so the results in the theorem 3.1 and the theorem 3.3 hold.

Let u $(\cdot, x)$=$u(\cdot)(x)$ and define f: J × X → X by f $(t,u) = \frac{\cos u(t,x)}{2+t}$. Set ω = 1, $L(t) = \frac{1}{2+t}$, $f(t,0) = \frac{1}{1+t}$, t ∈J. It is obvious to observe that

$$\|\frac{\cos u_1(t,x)}{2+t} - \frac{\cos u_2(t,x)}{2+t}\| \leq \frac{1}{2+t}\|u_1 - u_2\|,$$

and we can get $\frac{1}{2+t} \leq \frac{1}{2}$, $\|f(s,0)\| \leq \frac{1}{2}$.

Now all the assumptions of theorem 3.3 are satisfied, so the system has a unique solu- tion on $[0, T]$.

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