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An Analytical Exploration of Limits and Infinitesimals

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Abstract

Limits and infinitesimals are crucial components of calculus and advanced mathematics. This paper discusses the fundamental concepts and applications of limits, focusing on their significance in defining continuity, derivatives, and numerical representation techniques. We draw a parallel between infinitesimals and scientific notation to illustrate their role in facilitating complex calculations and simulating phenomena. Through analysis, we demonstrate how limits and infinitesimals provide a robust framework for understanding function behavior, explaining functions, and calculating limits.

Keywords

Limit; Infinitesimal; Order; Taylor's theorem.

1. INTRODUCTION

The development of calculus is inextricably linked to the concept of limits. Refining the limit concept has significant implications in the realm of calculus. Limits are also foundational to the theories of continuity, derivatives, and integration of functions. Infinitesimals are critical in the study of limits. By analyzing infinitesimals and their orders, we can identify the orders of infinitesimals for various functions. The order of infinitesimals gives theoretical support for approximate function calculations and quantitative dynamics of minute transformations. In complex model analysis, infinitesimals are indispensable. For instance, when examining numerical counting and size, scientific notation can represent numbers as powers of 10, simplifying counting and providing an initial analysis of numerical magnitude. Scientific notation simplifies comparisons between large and small numbers without extensive zeros. Infinitesimals[1-4] and order can be considered as the scientific notation of calculus. Despite lacking rigor, they offer convenient and fast function approximations and order calculations.

The standard definition of a limit is given as follows: For any $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. At this point, L is called the limit of $f(x)$ as x approaches a. From the definition, we observe that the concept of a limit fundamentally concerns the establishment of inequalities. It essentially involves transforming input conditions into the result of whether the inequality holds. An alternative equivalent definition of a limit can be given through transformation: For any given $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then the inequality $|f(x) - L| < \epsilon$ holds. Since the limit of $|f(x) - L|$ is 0, we can call it an infinitesimal. Therefore, the study of limits and infinitesimals is equivalent; it is essential to recognize that infinitesimals are variables, not constants (unlike zero, which can be

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seen as a constant function). We must treat infinitesimals as variables, not constant numbers in scientific notation.

2. INFINITESIMALS AND ORDER

2.1. Equivalent Infinitesimals

For the important limit $\lim_{x\to 0}\frac{\sin x}{x}=1$, we can represent it as $\sin x \sim x$, which means that as $x\to 0$, the values of the functions $\sin x$ and x are very close. From a quotient perspective, they are of the same order. We omit the rigorous proof here; readers can find it in any introductory calculus book. Generalizing this equivalence concept, for any two functions $f(x)$ and $g(x)$, if $\lim_{x\to a}\frac{f(x)}{g(x)}=A$, we denote it as $f(x)\sim Ag(x)$. Here,$-\infty < a < +\infty$.

This equivalence relation satisfies the following three properties:

1. Reflexivity: $\mathrm{f} \sim \mathrm{f}$
2. Symmetry: $\mathrm{f} \sim \mathrm{g} \Leftrightarrow \mathrm{g} \sim \mathrm{f}$
3. Transitivity: If $\mathrm{f} \sim \mathrm{g}$ and $\mathrm{g} \sim \mathrm{h}$, then $\mathrm{f} \sim \mathrm{h}$

2.2. Applications of Infinitesimals

The introduction of infinitesimals considerably facilitates the computation of function limits. For example, when dealing with inverse trigonometric and logarithmic functions, the inherent complexity of these functions can lead to significant computational difficulties. However, by expressing them as power functions, we can significantly reduce the computational complexity while still achieving excellent results. To illustrate this point with an example, consider the limit: $\lim_{x\to 0}\frac{\ln(1+x^3)}{\arctan x^3}$. First, according to the principle of equivalent infinitesimals, we have $\ln(1+x^3)\sim x^3 \sim \arctan x^3$. Therefore, $\lim_{x\to 0}\frac{\ln(1+x^3)}{\arctan x}=\lim_{x\to 0}\frac{x^3}{x^3}=1$.

Relying on a very simple conclusion: two functions $\ln(1+x^3)\sim x^3$, which are of the same order at $x=0$, then the original limit can be reformulated as a new ratio. Since we are trying to "simplify" the limit of a continuous function into a power function, we can use the power function to define the order of the limit. That is, if $\lim_{x\to x_0} f(x)=0$ and there exists a constant $\alpha>0$ such that $\lim_{x\to x_0}\frac{f(x)}{x^\alpha}=A\neq 0$, then $f(x)\sim A(x-a)^\alpha$, and $f(x)$ is said to have an infinitesimal of order α near $x=x_0$.

α is akin to the order of magnitude in scientific notation, allowing the original limit to be simplified into a new ratio. Since we are attempting to "simplify" limits, we can focus on the basic operations of limits. Logarithmic, trigonometric, and exponential functions are very common. Therefore, the introduction of power functions is to simplify complex calculations. Orders are a useful tool, but not all functions have orders. Some functions do not have orders, such as the function $f(x)=x\sin\frac{1}{x}$, which does not have an order near $x=0$. This is because the limit $\lim_{x\to 0}\frac{x\sin\frac{1}{x}}{x^\alpha}$ does not exist. Readers can discuss the size of α relative to 1 to verify the nonexistence of the limit.

2.3. Notations o and O

When comparing infinitesimals of different orders, the following definitions can be used: Suppose there are two infinitesimals α and β, which approach zero as some variable a tends to a certain value. We say that $f(x)$ is a higher-order infinitesimal compared to $g(x)$, denoted as $f(x)=o\big(g(x)\big)(x\to a)$, if $\lim_{x\to a}\frac{f(x)}{g(x)}=0$. In other words, if the absolute value of

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$f(x)$ is smaller than the absolute value of $g(x)$ as x approaches a, then $f(x)$ is of a higher order than $g(x)$. Another notation is defined as follows: $f(x) = O\big(g(x)\big)(x \to a)$ means that there exists a deleted neighborhood of a and a non-negative constant M such that $\left|\frac{f(x)}{g(x)}\right| \leq M$ holds in this neighborhood.

The concept of a "deleted neighborhood" is not complex. When dealing with limits, $f(x)$ and $g(x)$ may not have limits at a, so we discuss a small interval around a excluding a itself, which is $(a-\delta, a) \cup (a, a+\delta)$. This is the roughest estimate, but it is very convenient. Computing the limit of a ratio can be cumbersome, but knowing the ratio is bounded can reduce a lot of trouble.

These notations are widely used in mathematical analysis and other fields. By utilizing known limit ratios and properties, it is easier to derive the limits of unknown functions, thereby improving the efficiency and accuracy of analysis. The notations o and O are particularly suitable for expressing the relationship between a function and its approximate equivalent. They succinctly capture the essence of size comparison, enabling researchers to make informed decisions based on their understanding of function behavior.

For example, for $\alpha > 1$ we can write $x^\alpha = o(x)$. In the definition of higher-order infinitesimals, using the function $g(x)$ as a reference, the convergence rate of the function $f(x)$ is faster than that of $g(x)$. Higher-order infinitesimals are transitive, which facilitates the determination of the order of infinitesimals.

If $f(x) = o(1)(x \to a)$, then $f(x) \to 0$. Of course, using o notation in this case might seem awkward, but using 1 as the denominator is indeed not problematic. At this point, comparing the convergence rate with a constant function, the function either converges or does not converge.

In Fourier series, a class of functions with "moderate decay" is studied, requiring the function $f(x)$ to satisfy $f(x) = O\left(\frac{1}{1+|x|^\alpha}\right)$ where $\alpha > 1$. Investigating the previous function without an order, we find that $x \sin\frac{1}{x} = O(x)$. Thus, O notation can provide definitive results regarding the order of functions under specific conditions, which is beneficial in the study of infinitesimals that do not have an order.

In mathematical analysis, the notations o ,O ,$\sim$ are often used as "reference" symbols. The symbol "$\sim$" is used to abstractly specify the conclusions about the ratios and limits that a function satisfies, and to use these conclusions to represent another function, thereby indirectly calculating the limit of the original function. This method is akin to simplifying a problem to a precision of a few decimal places.

For example, suppose there is a sequence of functions $f_n(x)$, and as n approaches infinity, its limit is some function $f(x)$. If the ratio or limit conclusion between $f_n(x)$ and $f(x)$ can be determined, then the symbol "$\sim$" can be used to abstract these conclusions and represent $f(x)$. In this way, the limit of $f(x)$ can be indirectly calculated by studying the abstract conclusions, thereby simplifying the analysis of the original problem.

This method has wide applications in mathematical analysis and other fields, particularly in dealing with the limits of complex functions and sequences. By using the known ratios and properties of limits, it is easier to derive the limits of unknown functions, thus speeding up the problem-solving process and improving the efficiency and accuracy of the analysis.

3. TAYLOR'S THEOREM

The Taylor formula [5] can be understood as using the value of a function at a certain point to describe a function's value near that point. For sufficiently smooth functions (i.e., functions

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with sufficiently many continuous derivatives), the function's approximate structure near that point can be constructed based on the values of the function's derivatives at that point. The Taylor formula uses these derivative values as coefficients to construct a polynomial, called the Taylor polynomial, to approximate the function's values near that point. Taylor's theorem gives the remainder term, which represents the deviation between this polynomial and the actual function values.

Specifically, for a function $f(x)$ that has derivatives up to order n at x_0, its Taylor formula can be expressed as:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o((x - x_0)^n) \quad (1)$$

This theorem provides the order properties of a function near a certain point, which can be considered as a "few decimal places approximation" of the function compared to the equivalent infinitesimal. This approximation property greatly simplifies the process of handling limit problems. It eliminates the need to worry about the addition and subtraction between infinitesimals, as one can estimate the order of the denominator in advance and then use the Taylor formula to perform the corresponding approximation calculation for the numerator. Let's illustrate this with an example. See the following example:

Calculating the limit:

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 - \left(x - \frac{1}{3!}x^3\right)}{x^3} = \frac{1}{2} \quad (2)$$

In this problem, we can see that the denominator is a third-order infinitesimal. Therefore, based on the order of the denominator, it is sufficient to retain the numerator's terms up to the third order, because higher-order infinitesimals relative to the denominator can be ignored in limit calculations. This illustrates an important significance of the Taylor formula. To further study the properties of functions, the following quantitative Taylor's theorem is given:

If the function $f(x)$ has continuous derivatives up to order n at the point x_0, and has an $(n + 1)$-th order derivative within (a, b) then there exists at least one $\xi \in (a, b)$ such that:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \quad (3)$$

Here, $f^{(n)}(x_0)$ denotes the value of the n-th derivative of the function $f(x)$ at the point x_0. The characteristic of the Taylor series is that it has no remainder term; that is, for sufficiently smooth functions, the Taylor series can accurately represent the function within its interval of convergence. This series expansion gives a powerful tool that can be widely used in mathematical analysis, physics, engineering, and other fields. By studying Taylor series, one can gain a deeper understanding of the properties and behavior of functions, thereby promoting the development of mathematical theory and practical applications.

4. CONCLUSIONS

In this paper, we explored the conceptual relationship between infinitesimals and limits, introducing the notions of equivalent infinitesimals and same-order infinitesimals, which vastly facilitate the calculation of limits and the approximation of functions. While infinitesimals show advantages in approximation, they are not applicable for addition and subtraction operations. To delve deeper into the properties of functions, we studied higher-order methods: Taylor's formula and Taylor's theorem.

Taylor's formula is a method for high-order approximation of functions. It effectively describes the behavior of functions near a point. When the remainder term at that point is not needed, Taylor's formula is a concise and practical choice. Nevertheless, when higher-order remainder estimation and calculation are required, Taylor's formula still has limitations.

Taylor's theorem provides a more detailed and profound expansion of functions, satisfying various needs for understanding function properties, which is beyond the capability of infinitesimals alone and represents an extension of its concept. Additionally, Taylor's theorem offers strong polynomial-based support for function representation, providing a potent set of tools. Their application spans mathematics, engineering, physics, and other fields, offering significant support and guidance for research and practical work.

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